

Quantum Mechanics Lecture Notes

Chapter 1 The Wave Function

➤ Schrödinger equation

Classical mechanics:

$$x(t), \quad v = \frac{dx}{dt} \quad ; \quad \text{Newton's second law : } F = -\frac{\partial V}{\partial x} = m \frac{d^2x}{dt^2}$$

Quantum mechanics :

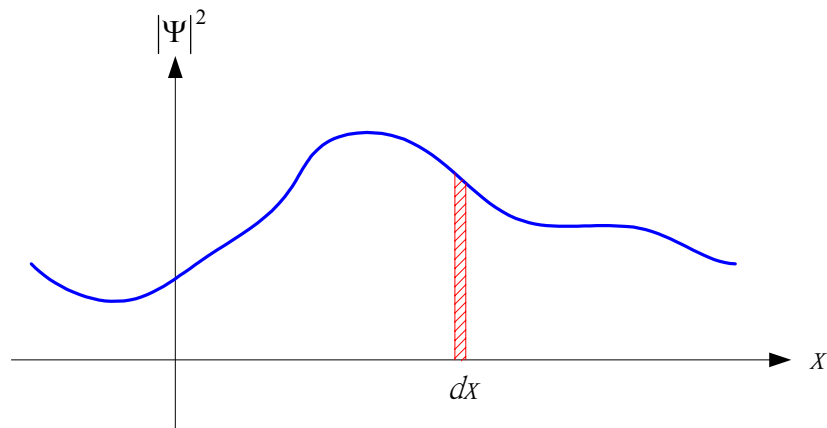
wave function $\Psi(x, t)$;

$$\text{time-depended Schrödinger equation : } i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi$$

➤ Statistical interpretation

Born's statistical interpretation :

$$|\Psi(x, t)|^2 dx = \{ \text{probability of finding the particle between } x \text{ and } (x+dx) \text{ at time } t \}$$



➤ Probability

$$|\Psi(x, t)|^2 = \Psi^* \Psi \quad \text{is probability density}$$

$$\text{the probability of infinite interval : } \int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx$$

➤ Normalization

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$$

- (i) If $\Psi(x,t)$ is a solution, $A\Psi(x,t)$ is also a solution.
Normalized the wave function to determine the factor A
- (ii) If above the integral is infinite for some wave functions,
no factor to make it be normalizable.
The non-normalizable wave function cannot represent particles.
- (iii) the condition of wave function which can be normalizable

$$\Psi(x,t) = 0 \quad \text{as } x \rightarrow \pm \infty$$

Deduce:

$$\frac{d}{dt} \int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} |\Psi(x,t)|^2 dx$$

$$\frac{\partial}{\partial t} |\Psi(x,t)|^2 = \Psi^* \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t} \Psi$$

from Schrödinger equation $\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V\Psi$ and $\frac{\partial \Psi^*}{\partial t} = \frac{-i\hbar}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V\Psi^*$

$$\rightarrow \frac{\partial}{\partial t} |\Psi(x,t)|^2 = \frac{i\hbar}{2m} \left(\Psi^* \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial x^2} \Psi \right) = \frac{\partial}{\partial x} \left[\frac{i\hbar}{2m} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) \right]$$

$$\rightarrow \frac{d}{dt} \int_{-\infty}^{\infty} |\Psi|^2 dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left[\frac{i\hbar}{2m} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) \right] dx = \frac{i\hbar}{2m} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) \Big|_{-\infty}^{\infty}$$

If wave function can be normalizable then above form must be zero

So $\Psi(x,t) = 0$ as $x \rightarrow \pm \infty$ and if Ψ is normalized at $t = 0$,

it stays normalized for all future time

➤ **Operator and expectation value (average or mean)**

expectation value of position x : $\langle x \rangle = \int_{-\infty}^{\infty} x |\Psi(x, t)|^2 dx = \int_{-\infty}^{\infty} \Psi^* x \Psi dx$

expectation value of momentum $\hat{p}$: $\langle \hat{p} \rangle = m \frac{d\langle x \rangle}{dt} = \int_{-\infty}^{\infty} \Psi^* (-i\hbar \frac{\partial}{\partial x}) \Psi dx$

operator x represent position; **operator** $-i\hbar \frac{\partial}{\partial x}$ represent momentum in x -direction

all physics quantities can be written in terms of position and momentum

$$\langle Q(x, \hat{p}) \rangle = \int \Psi^* Q(x, -i\hbar \frac{\partial}{\partial x}) \Psi dx$$

➤ **Heisenberg uncertainty principle**

$$\boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}} \quad (\text{the proof in chapter 3})$$

σ : **standard deviation**

$\sigma^2 \equiv \langle (\Delta j)^2 \rangle$: the **variance** of distribution

where $\Delta j = j - \langle j \rangle$

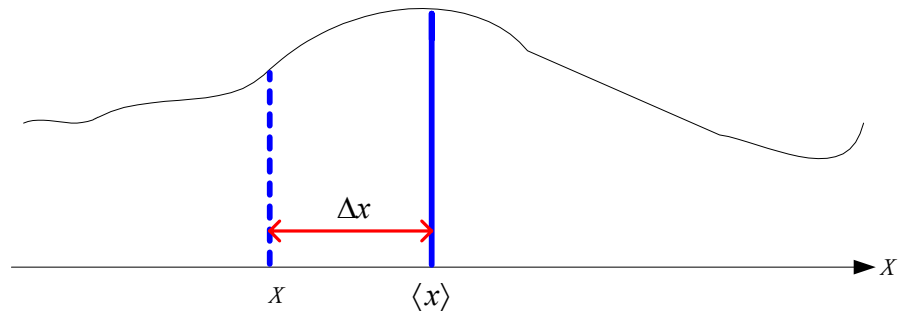
j : individual physics quantity

$$\begin{aligned} \sigma^2 &= \langle (\Delta j)^2 \rangle = \langle (j - \langle j \rangle)^2 \rangle \\ &= \langle (j^2 - 2\langle j \rangle \langle j \rangle + \langle j \rangle^2) \rangle \\ &= \langle j^2 \rangle - 2\langle j \rangle \langle j \rangle + \langle j \rangle^2 \\ &= \langle j^2 \rangle - \langle j \rangle^2 \end{aligned}$$

so σ_x : standard deviation in position

σ_p : standard deviation in momentum

$$\sigma_x^2 = \langle (\Delta x)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 \quad \sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2$$



$$\Delta x = x - \langle x \rangle$$

$$\sigma_x^2 \equiv \langle (\Delta x)^2 \rangle$$