

Chapter 3 Formalism in Hilbert Space

3.1 Linear algebra

➤ Vectors

vectors : $|\alpha\rangle, |\beta\rangle, |\gamma\rangle, \dots$

commutative of addition : $|\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle$

associative of addition : $|\alpha\rangle + (|\beta\rangle + |\gamma\rangle) = (|\alpha\rangle + |\beta\rangle) + |\gamma\rangle$

scalar multiplication : $a|\alpha\rangle = |\gamma\rangle$, a is scalar

distributive : $a(|\alpha\rangle + |\beta\rangle) = a|\alpha\rangle + a|\beta\rangle$
 $(a+b)|\alpha\rangle = a|\alpha\rangle + b|\alpha\rangle$

associative of multiplication : $a(b|\alpha\rangle) = (ab)|\alpha\rangle$

➤ Inner product

the inner product of two vectors $|\alpha\rangle, |\beta\rangle$ is $\langle\alpha|\beta\rangle$, and

$$\begin{aligned}\langle\alpha| &= |\alpha\rangle^* \\ \langle\alpha|\beta\rangle &= \langle\beta|\alpha\rangle^* \\ \langle\alpha|\alpha\rangle &\geq 0, \text{ and } \langle\alpha|\alpha\rangle = 0 \Leftrightarrow |\alpha\rangle = |0\rangle \\ \langle\alpha|(b|\beta\rangle + c|\gamma\rangle) &= b\langle\alpha|\beta\rangle + c\langle\alpha|\gamma\rangle, \quad b, c : \text{constant} \\ \langle\alpha|c\beta\rangle &= c\langle\alpha|\beta\rangle, \quad \langle c\alpha|\beta\rangle = c^*\langle\alpha|\beta\rangle\end{aligned}$$

norm of the vector :

$$\|\alpha\| = \sqrt{\langle\alpha|\alpha\rangle}$$

orthonormal :

$$\langle\alpha_i|\alpha_j\rangle = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Schwarz inequality :

$$|\langle\alpha|\beta\rangle|^2 \leq \langle\alpha|\alpha\rangle \langle\beta|\beta\rangle$$

$$\text{and angle between } |\alpha\rangle, |\beta\rangle \text{ is : } \cos\theta = \sqrt{\frac{\langle\alpha|\beta\rangle \langle\beta|\alpha\rangle}{\langle\alpha|\alpha\rangle \langle\beta|\beta\rangle}}$$

vector space with an inner product is called **inner product space**

➤ **Linear transformations**

$$|\alpha\rangle \rightarrow |\alpha'\rangle = T|\alpha\rangle$$

$$\mathbf{T}(a|\alpha\rangle + b|\beta\rangle) = a(\mathbf{T}|\alpha\rangle) + b(\mathbf{T}|\beta\rangle)$$

for any vectors $|\alpha\rangle, |\beta\rangle$, any scalars a, b

$\mathbf{T}$ is linear transformation

the **matrix** form $\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix}$

the **transpose** $\tilde{\mathbf{T}} = \begin{bmatrix} T_{11} & T_{21} & \cdots & T_{n1} \\ T_{12} & T_{22} & \cdots & T_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ T_{1n} & T_{2n} & \cdots & T_{nn} \end{bmatrix}$

a square matrix $\mathbf{T}$ is **symmetric** if $\tilde{\mathbf{T}} = \mathbf{T}$
and is **antisymmetric** if $\tilde{\mathbf{T}} = -\mathbf{T}$

$$(\tilde{\mathbf{S}}\mathbf{T}) = \tilde{\mathbf{T}}\tilde{\mathbf{S}}$$

complex conjugate of a matrix $\mathbf{T}$ is

$$\mathbf{T}^* = \begin{bmatrix} T_{11}^* & T_{12}^* & \cdots & T_{1n}^* \\ T_{21}^* & T_{22}^* & \cdots & T_{2n}^* \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1}^* & T_{n2}^* & \cdots & T_{nn}^* \end{bmatrix}$$

Hermitian conjugate of a matrix $\mathbf{T}$ is

$$\mathbf{T}^+ \equiv \tilde{\mathbf{T}}^* = \begin{bmatrix} T_{11}^* & T_{21}^* & \cdots & T_{n1}^* \\ T_{12}^* & T_{22}^* & \cdots & T_{n2}^* \\ \vdots & \vdots & \ddots & \vdots \\ T_{1n}^* & T_{2n}^* & \cdots & T_{nn}^* \end{bmatrix}$$

a square matrix **T** is **Hermitian** (or **self-adjoint**) if $\mathbf{T}^+ = \mathbf{T}$
a square matrix **T** is **skew Hermitian** if $\mathbf{T}^+ = -\mathbf{T}$
a square matrix **T** is **unitary** if $\mathbf{T}^+ = \mathbf{T}^{-1}$

in general , $\mathbf{ST} \neq \mathbf{TS}$

define **commutator** $[\mathbf{S}, \mathbf{T}] = \mathbf{ST} - \mathbf{TS}$

$$(\mathbf{ST})^+ = \mathbf{T}^+ \mathbf{S}^+$$

➤ Eigenvalues and eigenvectors

for satisfy $\mathbf{T} |\alpha\rangle = \lambda |\alpha\rangle$

λ is eigenvalue corresponding to eigenvector $|\alpha\rangle$

➡ $(\mathbf{T} - \lambda \mathbf{1})|\alpha\rangle = \mathbf{0}$ $\mathbf{0}$ is **zero matrix**

$$\mathbf{1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

➡ $\det(\mathbf{T} - \lambda \mathbf{1}) = \begin{vmatrix} T_{11} - \lambda & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} - \lambda & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} - \lambda \end{vmatrix} = 0$

➡ $C_n \lambda^n + C_{n-1} \lambda^{n-1} + \dots + C_1 \lambda + C_0 = 0$ is **characteristic equation**

➤ Hermitian transformations

T is Hermitian , $\mathbf{T}^+ = \mathbf{T}$

and $\langle \mathbf{T}^+ \alpha | \beta \rangle = \langle \mathbf{T} \alpha | \beta \rangle = \langle \alpha | \mathbf{T} \beta \rangle$ is Hermitian transformations

three properties :

(i) eigenvalues of a Hermitian transformation are real

let λ is an eigenvalue of **T** : $\mathbf{T} |\alpha\rangle = \lambda |\alpha\rangle$, $|\alpha\rangle \neq 0$

$$\rightarrow \begin{cases} \langle \alpha | T\alpha \rangle = \langle T\alpha | \alpha \rangle = \langle \lambda\alpha | \alpha \rangle = \lambda^* \langle \alpha | \alpha \rangle \\ \langle \alpha | T\alpha \rangle = \langle \alpha | \lambda\alpha \rangle = \lambda \langle \alpha | \alpha \rangle \end{cases}$$

$$\rightarrow (\lambda - \lambda^*)(\langle \alpha | \alpha \rangle) = 0 \quad , \quad \text{and} \quad \langle \alpha | \alpha \rangle \neq 0$$

$$\text{so} \quad \lambda = \lambda^* \quad , \quad \lambda \text{ is real}$$

(ii) eigenvectors of a Hermitian transformation belonging to distinct eigenvalues are orthogonal

$$T|\alpha_1\rangle = \lambda_1|\alpha_1\rangle \quad , \quad T|\alpha_2\rangle = \lambda_2|\alpha_2\rangle \quad , \quad \lambda_1 \neq \lambda_2$$

$$\rightarrow \begin{cases} \langle \alpha_1 | T\alpha_2 \rangle = \langle \alpha_1 | \lambda_2\alpha_2 \rangle = \lambda_2 \langle \alpha_1 | \alpha_2 \rangle \\ \langle \alpha_1 | T\alpha_2 \rangle = \langle T\alpha_1 | \alpha_2 \rangle = \langle \lambda_1\alpha_1 | \alpha_2 \rangle = \lambda_1^* \langle \alpha_1 | \alpha_2 \rangle = \lambda_1 \langle \alpha_1 | \alpha_2 \rangle \end{cases}$$

$$\rightarrow (\lambda_1 - \lambda_2)(\langle \alpha_1 | \alpha_2 \rangle) = 0 \quad ,$$

$$\rightarrow \lambda_1 \neq \lambda_2 \quad , \quad \text{so} \quad \langle \alpha_1 | \alpha_2 \rangle = 0 \quad , \quad |\alpha_1\rangle \quad , \quad |\alpha_2\rangle \text{ are orthogonal}$$

(iii) a set orthonormal eigenvectors of a Hermitian transformation span the space

$$|\beta\rangle = \sum a_i |\alpha_i\rangle$$

3.2 Function space

the inner product of functions $f(x)$, $g(x)$ is defined by **Dirac notation**

$$\langle f | g \rangle = \int_a^b f^*(x)g(x)dx$$

$$\begin{aligned} \text{and} \quad \langle f | g \rangle &= \langle g | f \rangle^* \\ \langle f | f \rangle &\geq 0 \quad , \quad \langle f | f \rangle = 0 \Leftrightarrow |f\rangle = 0 \\ \langle f | ag \rangle &= a \langle f | g \rangle \\ \langle af | g \rangle &= a^* \langle f | g \rangle \quad , \quad a \text{ is scalar} \\ \langle f | (a|g\rangle + b|h\rangle) &= a \langle f | g \rangle + b \langle f | h \rangle \end{aligned}$$

above the inner product must *converge* , every admissible function f must be **square integrable**

$$\int |f(x)|^2 dx < \infty$$

➤ Hermitian operators

$$\begin{aligned}\hat{T} f(x) &= \lambda f(x) & \hat{T} &: \text{operator} \\ \lambda &: \text{eigenvalue} \\ f(x) &: \text{eigenfunction of operator}\end{aligned}$$

$$\begin{aligned}\hat{T}(a|\alpha\rangle + b|\beta\rangle) &= a(\hat{T}|\alpha\rangle) + b(\hat{T}|\beta\rangle) \\ &\text{for any vectors } |\alpha\rangle, |\beta\rangle, \text{ any scalars } a, b\end{aligned}$$

$\hat{T}$ is linear transformation

Hermitian operator is a linear transformation operator

$$\text{and } \langle f | \hat{T} g \rangle = \langle \hat{T} f | g \rangle \quad \text{or} \quad \int f^* \hat{T} g dx = \int (\hat{T} f)^* g dx$$

example

1. momentum operator $\hat{p}$ is Hermitian

$$\begin{aligned}\langle \Psi | \hat{p} \Psi \rangle &= \int \Psi^* (-i\hbar \frac{d}{dx}) \Psi dx = -i\hbar \left[(\Psi^* \Psi)_{-\infty}^{\infty} - \int \Psi \frac{d}{dx} \Psi^* dx \right] \\ &= \int \Psi (-i\hbar \frac{d}{dx} \Psi)^* dx = \langle \hat{p} \Psi | \Psi \rangle\end{aligned}$$

2. Hamiltonian operator $\hat{H}$ is Hermitian

$$\begin{aligned}\langle \Psi | \hat{H} \Psi \rangle &= \langle \Psi | (-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V) \Psi \rangle = \langle \Psi | (-\frac{\hbar^2}{2m} \frac{d^2}{dx^2}) \Psi \rangle + V \langle \Psi | \Psi \rangle \\ &= -\frac{\hbar^2}{2m} \int \Psi^* \frac{d^2}{dx^2} \Psi dx + V \int |\Psi|^2 dx \\ &= -\frac{\hbar^2}{2m} \left[\left(\Psi^* \frac{d\Psi}{dx} \right)_{-\infty}^{\infty} - \int \frac{d\Psi}{dx} \frac{d\Psi^*}{dx} dx \right] + V \int |\Psi|^2 dx \\ &= \frac{\hbar^2}{2m} \left[\left(\Psi \frac{d\Psi^*}{dx} \right)_{-\infty}^{\infty} - \int \Psi \frac{d^2\Psi^*}{dx^2} dx \right] + V \int |\Psi|^2 dx \\ &= \int \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V \right) \Psi^* \Psi dx = \int \hat{H} \Psi^* \Psi dx \\ &= \int (\hat{H} \Psi)^* \Psi dx = \langle \hat{H} \Psi | \Psi \rangle\end{aligned}$$

Hermitian operator character :

- (i) An hermitian operator is respect to a set of functions ,
but not respect to another set
- (ii) An hermitian operator is respect to a set of functions over a range of variable,
but not respect to the same set over a different range
- (iii) Product of two hermitian operators may or may not be hermitian

$$\hat{A}\hat{B} \text{ is hermitian } \Leftrightarrow \hat{A}, \hat{B} \text{ commute}$$
- (iv) The eigenfunctions of a hermitian operator are complete
- (v) The eigenvalues of a hermitian operator are real

➤ Eigenfunction expansions

- (i) a set of orthonormal eigenfunctions Ψ_i of a hermitian operator

$$f = \sum a_i \Psi_i \dots\dots\dots(3.1)$$

the summation in equation (3.1) converges to f if the set of eigenfunction is complete

complete : no other function g exists such that $\langle g | \Psi_i \rangle = 0$

determine a_i :

$$\langle \Psi_j | f \rangle = \langle \Psi_j | \sum a_i \Psi_i \rangle = \sum a_i \langle \Psi_j | \Psi_i \rangle = a_j$$

$$\text{so } a_i = \langle \Psi_i | f \rangle$$

- (ii) suppose Ψ_i are simultaneously eigenfunctions of two hermitian operator

$\hat{A}, \hat{B}$ with eigenvalues α_i, β_i

$$\begin{aligned} \begin{cases} \hat{A}\Psi_i = \alpha_i\Psi_i \\ \hat{B}\Psi_i = \beta_i\Psi_i \end{cases} & \xrightarrow{\text{blue arrow}} \begin{cases} \hat{B}\hat{A}\Psi_i = \alpha_i\hat{B}\Psi_i = \alpha_i\beta_i\Psi_i \\ \hat{A}\hat{B}\Psi_i = \beta_i\hat{A}\Psi_i = \alpha_i\beta_i\Psi_i \end{cases} \\ & \xrightarrow{\text{blue arrow}} (\hat{A}\hat{B} - \hat{B}\hat{A})\Psi_i = [\hat{A}, \hat{B}]\Psi_i = 0, \Psi_i \neq 0 \\ & \xrightarrow{\text{blue arrow}} [\hat{A}, \hat{B}] = 0 \end{aligned}$$

so operators $\hat{A}, \hat{B}$ have the same complete set of eigenfunctions

then $\hat{A}, \hat{B}$ **commute**

3.3 Hilbert space

a **complete linear** vector space of infinite dimensionality is **Hilbert space**

(i) **ket** vectors (or kets)

the basis vectors in Hilbert space and determine a *ket space*

Dirac notation symbol : $|\Psi_i\rangle$ or $|i\rangle$

$$\hat{A}|\Psi_i\rangle = |\hat{A}\Psi_i\rangle$$

$$\text{if } \hat{A}|\Psi_i\rangle = |\hat{A}\Psi_i\rangle = \alpha_i |\Psi_i\rangle$$

$$|\Psi_i\rangle \text{ are eigenkets of operator } \hat{A}$$

(ii) **bra** vectors (or bras)

vectors determine a *bra space*

Dirac notation symbol : $\langle\Psi_i|$ or $\langle i|$

$$\langle\hat{A}\Psi_i| = \langle\Psi_i|\hat{A}^+$$

(iii) every ket in ket space corresponding a bra in bra space

$$\begin{array}{ccc} |\Psi_i\rangle & \longleftrightarrow & \langle\Psi_i| \\ |c\Psi_i\rangle & \longleftrightarrow & \langle c\Psi_i| \end{array}$$

$$\text{the relation is : } |\Psi_i\rangle^+ = \langle\Psi_i|, \quad \langle\Psi_i|^+ = |\Psi_i\rangle$$

$$\text{so } \langle c\Psi_i| = (|c\Psi_i\rangle)^+ = (c|\Psi_i\rangle)^+ = c^*(|\Psi_i\rangle)^+ = c^*\langle\Psi_i|$$

(iv)

the inner product of $\langle\Psi_j|$ and $|\hat{A}\Psi_i\rangle$ in Dirac notation is

$$\langle\Psi_j|\hat{A}|\Psi_i\rangle = \langle\Psi_j|\hat{A}\Psi_i\rangle$$

$$\langle\hat{A}\Psi_j|\Psi_i\rangle = \langle\Psi_j|\hat{A}^+|\Psi_i\rangle = \langle\hat{A}^+\Psi_j|\Psi_i\rangle^*$$

$$= \langle\Psi_j|\hat{A}\Psi_i\rangle^*$$

the outer product of $\langle\Psi_j|$, $|\Psi_i\rangle$ is $|\Psi_i\rangle\langle\Psi_j|$

$$\text{and } (|\Psi_i\rangle\langle\Psi_j|)^+ = |\Psi_j\rangle\langle\Psi_i|$$

The quantum mechanics formalism in Hilbert space

- (1) The state of quantum system is represented by a normalized vector $(|\psi_i\rangle)$ in the Hilbert space L_2
- (2) Any observable quantities $Q(x, p, t)$ are represented by Hermitian operator $\hat{Q}(x, -i\hbar \frac{\partial}{\partial x}, t)$, the expectation value of Q in the state $|\psi\rangle$ is $\langle\psi|\hat{Q}|\psi\rangle$
- (3) If you measure an observable Q of a quantum system, you are certain to get one of the eigenvalue of $\hat{Q}$, $\hat{Q}|e_n\rangle = \lambda_n |e_n\rangle$, $\langle e_n | e_m \rangle = \delta_{nm}$
 $|\psi\rangle = \sum c_n e_n$, $c_n = \langle e_n | \psi \rangle$, $|c_n|^2 = |\langle e_n | \psi \rangle|^2$

3.4 Uncertainty principle

➤ Generalized uncertainty principle

for any observable $\hat{A}$,

$$\sigma_A^2 = \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle = \langle \Psi | (\hat{A} - \langle \hat{A} \rangle)^2 | \Psi \rangle = \langle (\hat{A} - \langle \hat{A} \rangle) \Psi | (\hat{A} - \langle \hat{A} \rangle) \Psi \rangle$$

$$\text{let } |f\rangle \equiv (\hat{A} - \langle \hat{A} \rangle) | \Psi \rangle, \quad \langle f | \equiv \langle (\hat{A} - \langle \hat{A} \rangle) \Psi |$$

$$\rightarrow \sigma_A^2 = \langle f | f \rangle$$

for any another observable B , $\sigma_B^2 = \langle g | g \rangle$, $|g\rangle \equiv (\hat{B} - \langle \hat{B} \rangle) | \Psi \rangle$

By Schwarz inequality :

$$\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2$$

let complex $z = \langle f | g \rangle$, $|z|^2 = (\text{Re}(z))^2 + (\text{Im}(z))^2 \geq (\text{Im}(z))^2 = \left[\frac{1}{2i} (z - z^*) \right]^2$

$$\rightarrow \sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} [\langle f | g \rangle - \langle g | f \rangle] \right)^2$$

$$\text{and } \langle f | g \rangle = \langle (\hat{A} - \langle \hat{A} \rangle) \Psi | (\hat{B} - \langle \hat{B} \rangle) \Psi \rangle = \langle \Psi | (\hat{A} - \langle \hat{A} \rangle)(\hat{B} - \langle \hat{B} \rangle) \Psi \rangle$$

$$\begin{aligned}
&= \langle \Psi | (\hat{A}\hat{B} - \hat{A}\langle\hat{B}\rangle - \hat{B}\langle\hat{A}\rangle + \langle\hat{A}\rangle\langle\hat{B}\rangle) \Psi \rangle \\
&= \langle \Psi | \hat{A}\hat{B}\Psi \rangle - \langle\hat{B}\rangle \langle \Psi | \hat{A}\Psi \rangle - \langle\hat{A}\rangle \langle \Psi | \hat{B}\Psi \rangle + \langle\hat{A}\rangle\langle\hat{B}\rangle \langle \Psi | \Psi \rangle \\
&= \langle \hat{A}\hat{B} \rangle - \langle\hat{B}\rangle\langle\hat{A}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle + \langle\hat{A}\rangle\langle\hat{B}\rangle \\
&= \langle \hat{A}\hat{B} \rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle
\end{aligned}$$

similarly $\langle g | f \rangle = \langle \hat{B}\hat{A} \rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$

$$\begin{aligned}
\rightarrow \quad \langle f | g \rangle - \langle g | f \rangle &= \langle \hat{A}\hat{B} \rangle - \langle \hat{B}\hat{A} \rangle = \langle [\hat{A}, \hat{B}] \rangle \\
&= \langle \hat{A}\hat{B} - \hat{B}\hat{A} \rangle
\end{aligned}$$

$$\text{so } \sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

example

Heisenberg uncertainty principle :
$$\begin{cases} \hat{A} = x \\ \hat{B} = \hat{p} = -i\hbar \frac{d}{dx} \end{cases}$$

$$\begin{aligned}
\text{then } [\hat{x}, \hat{p}]f(x) &= x(-i\hbar) \frac{df}{dx} - (-i\hbar) \frac{d}{dx}(xf) \\
&= -i\hbar x \frac{df}{dx} + i\hbar x \frac{df}{dx} + i\hbar f \\
&= i\hbar f
\end{aligned}$$

so $[\hat{x}, \hat{p}] = i\hbar$

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{1}{2i} i\hbar \right)^2 = \left(\frac{\hbar}{2} \right)^2 \quad \rightarrow \quad \sigma_x \sigma_p \geq \frac{\hbar}{2}$$

any pair of observables whose corresponding operators do not commute represents that they are **incompatible observables**

➤ Energy-time uncertainty principle

for the expectation value of an observable $Q(x, p, t)$

$$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{d}{dt}\langle\Psi|\hat{Q}|\Psi\rangle = \langle\frac{\partial\Psi}{\partial t}|\hat{Q}|\Psi\rangle + \langle\Psi|\frac{\partial\hat{Q}}{\partial t}|\Psi\rangle + \langle\Psi|\hat{Q}|\frac{\partial\Psi}{\partial t}\rangle$$

by Schrödinger equation $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$

$$\begin{aligned}\rightarrow \frac{d}{dt}\langle\hat{Q}\rangle &= \langle-\frac{i}{\hbar}\hat{H}\Psi|\hat{Q}|\Psi\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle + \langle\Psi|(-\frac{i}{\hbar})\hat{Q}\hat{H}|\Psi\rangle \\ &= \frac{i}{\hbar}\langle\hat{H}\Psi|\hat{Q}|\Psi\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle - \frac{i}{\hbar}\langle\Psi|\hat{Q}\hat{H}|\Psi\rangle \\ &= \frac{i}{\hbar}\langle\Psi|\hat{H}\hat{Q}|\Psi\rangle - \frac{i}{\hbar}\langle\Psi|\hat{Q}\hat{H}|\Psi\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle \\ &= \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle\end{aligned}$$

$$\rightarrow \langle[\hat{H},\hat{Q}]\rangle = \frac{\hbar}{i}\frac{d}{dt}\langle\hat{Q}\rangle - \hbar\langle\frac{\partial\hat{Q}}{\partial t}\rangle$$

typical case

operator $\hat{Q}$ is independent of t

$$\text{so } \sigma_H^2\sigma_Q^2 \geq \left(\frac{1}{2i}\langle[\hat{H},\hat{Q}]\rangle\right)^2 = \left(\frac{1}{2i}\hbar\frac{d\langle\hat{Q}\rangle}{dt}\right)^2 = \left(\frac{\hbar}{2}\right)^2\left(\frac{d\langle\hat{Q}\rangle}{dt}\right)^2$$

$$\rightarrow \sigma_H\sigma_Q \geq \frac{\hbar}{2}\left|\frac{d\langle\hat{Q}\rangle}{dt}\right|$$

$$\text{define } \Delta E \equiv \sigma_H, \quad \Delta t \equiv \frac{\sigma_Q}{\left|d\langle\hat{Q}\rangle/dt\right|}$$

$$\text{then } \Delta E\Delta t \geq \frac{\hbar}{2} \dots\dots\dots \text{ is energy-time uncertainty principle}$$