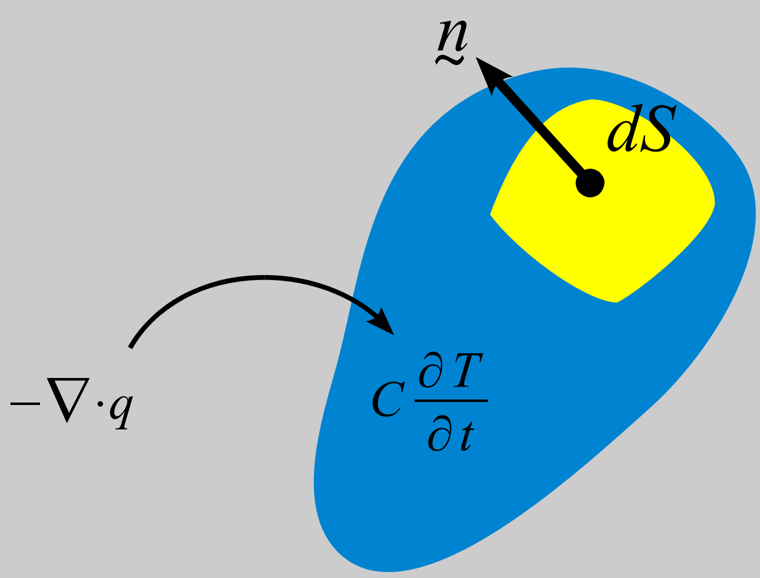
擴散方程式



$$\oint q\cdot n\,dS = \int \nabla\cdot q\, d\upsilon$$

$$-\int \nabla\cdot q\, d\upsilon = \int \rho\, c\,\frac{\partial T}{\partial t}\, d\upsilon$$

$$q = -k\nabla T \quad \text{(擴散定律)}$$

$$\Rightarrow \int \left(k\nabla^2 T - \rho\, c\,\frac{\partial T}{\partial t}\right) d\upsilon = 0$$

$$\Rightarrow \frac{\partial T}{\partial t} = \overbrace{\frac{k}{\rho\, c}}^{\alpha^2}\nabla^2 T$$

分離變數法

$$\nabla^2 T = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}$$

$$u \equiv \Psi(r)T(t)$$

$$\Rightarrow \frac{1}{\Psi}\nabla^2\Psi = \frac{1}{\alpha^2}T\frac{dT}{dt} = -\lambda^2$$

$$\Rightarrow \begin{cases} \dfrac{dT}{dt} + (\lambda\alpha)^2 T = 0 \longrightarrow T = e^{-(\lambda\alpha)^2 t} \\ (\nabla^2 + \lambda^2)\Psi \equiv 0 \end{cases}$$

正交函數展開法

卡氏座標系

$$\frac{\partial^2 \Psi}{\partial x^2}+\frac{\partial^2 \Psi}{dy^2}+\frac{\partial^2 \Psi}{dz^2}+\lambda^2\Psi=0$$

$$\Psi \equiv R(r)\Theta(\theta)\Phi(\phi)$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}+\lambda^2=0$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2}=-\gamma^2 \quad \rightarrow \quad Z=e^{\pm i\gamma^z}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2}=-\upsilon^2 \quad \rightarrow \quad Y=e^{\pm i\upsilon^y}(\mu^2+\upsilon^2+\gamma^2=\lambda^2)$$

$$\frac{1}{X}\frac{d^2X}{dx^2}=-\mu^2 \quad \rightarrow \quad X=e^{\pm i\mu^x}$$

$$\Rightarrow \Psi(x,\ y,\ z)=\sum Ae^{\pm i\mu^x \pm i\upsilon^y \pm ir^z},\ (\mu^2+\upsilon^2+\gamma^2=\lambda^2)$$

柱座標系

$$\nabla^2 \Psi \equiv \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial \Psi}{\partial \theta^2} + \frac{\partial^2 \Psi}{\partial z^2} + \lambda^2 \Psi = 0$$

$$\Psi \equiv R(r)\Theta(\theta)\Phi(\phi)$$

$$\Rightarrow \frac{1}{R}\frac{d^2 R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \frac{1}{r^2 \Theta}\frac{d^2 \Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2 Z}{d^2 z} + \lambda^2 = 0$$

$$\begin{cases} \dfrac{1}{Z}\dfrac{d^2 Z}{dz^2} = m^2 \quad \rightarrow \quad Z = e^{\pm mz} \\ \dfrac{1}{\Theta}\dfrac{d^2 \Theta}{d\theta^2} = -n^2 \quad \rightarrow \quad \Theta = e^{\pm i n \theta} \end{cases}$$

$$\Rightarrow \frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + (\lambda^2 + m^2 - \frac{n^2}{r^2})R = 0 \quad \rightarrow \quad R = J_{\pm n}(\sqrt{\lambda^2 + m^2}\, r)$$

$$\Rightarrow \Psi \equiv R\Theta Z = \sum A J_{\pm n}(\sqrt{\lambda^2 + m^2}\, r) e^{\pm i n \theta \pm mz}$$

球座標系

$$(\nabla^2+\lambda^2)\Psi \equiv \frac{\partial^2 \Psi}{\partial r^2}+\frac{2}{r}\frac{\partial \Psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial \Psi}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 \Psi}{\partial \phi^2}+\lambda^2\Psi = 0$$

$$\Psi(r,\ \theta,\ \phi) \equiv R(r)\Theta(\theta)\Phi(\phi)$$

$$\Rightarrow \frac{1}{R}\left[\frac{d^2 R}{dr^2}+\frac{2}{r}\frac{dR}{dr}\right]+\frac{1}{r^2\Theta}\left[\frac{d^2\Theta}{d\theta^2}+\cot\theta\frac{d\Theta}{d\theta}\right]+\frac{1}{r^2\sin^2\theta}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+\lambda^2 = 0$$

$$\Rightarrow \begin{cases} \dfrac{1}{\Phi}\dfrac{d^2\Phi}{d\phi^2} = -m^2 & \rightarrow \quad \Phi = e^{im\Phi} \\ \dfrac{1}{\Theta}\left[\dfrac{d^2\Theta}{d\theta^2}+\cot\theta\dfrac{d\Theta}{d\theta}\right] = \dfrac{m^2}{\sin^2\theta}-n(n+1) & \rightarrow \quad \Theta = CP_n^m(\cos\theta)+DQ_n^m(\cos\theta) \\ \dfrac{1}{R}\left[\dfrac{d^2R}{dr^2}+\dfrac{2}{r}\dfrac{dR}{dr}\right] = \dfrac{n(n+1)}{r^2}-\lambda^2 & \rightarrow \quad R = r^{-1/2}\left[AJ_{n+1/2}(\lambda r)+BY_{n+1/2}(\lambda r)\right] \end{cases}$$

$$\Rightarrow \Psi \equiv R\Theta\Phi = \sum r^{-1/2}\left[AJ_{n+1/2}(\lambda)+BY_{n+1/2}(\lambda r)\right]\left[CP_n^m(\cos\theta)+DQ_n^m(\cos\theta)\right]e^{\pm im\phi}$$