

Rupture theory of thin power-law liquid film

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The nonlinear rupture problem of a thin power-law liquid film on a horizontal plane is studied. A nonlinear evolution equation is first derived for $h(x,t)$, the film thickness, and then it is solved by numerical methods. The results reveal that the rupture time of the film decreases with decreasing the magnitude of the power-law exponent, n .

Research of the rupture of thin liquid films has been motivated by industrial applications in disperse and colloid system on the one hand, and the understanding of diverse biological phenomena on the other.¹⁻⁵ Ruckenstein and Jain⁶ studied the spontaneous rupture of a liquid film on a solid plane. They modeled the liquid as a Navier-Stokes continuum having an extra body force due to van der Waals attractions and used a lubrication approximation to obtain linear dynamic instability results. From these results we can obtain a rough estimate of the required time for the film to attain zero thickness at some point. Williams and Davis⁷ extended the analysis into the nonlinear regions. They formulated a long-wave theory to derive a nonlinear evolution equation for the film thickness and solved the nonlinear evolution equation numerically by treating it as an initial-value problem with periodic boundary conditions. Their results indicate that the nonlinearities will accelerate the rupture process. Furthermore, Reisfeld and Bankoff⁸ and Hwang *et al.*⁹ have extended the analysis by including a temperature-dependent viscosity model. They rescale the time variable by a suitable factor and show that the nonlinear evolution equation can be converted into a constant-viscosity equation. They found that with a cooling (heating) from the wall the rupture time will increase (reduce).

Apparently, most of the above studies¹⁻⁷ were addressed based on the Newtonian fluid model. However, the rheological behavior of the fluid during the manufacturing process or the glue in biological chemistry cannot be completely described by the Newtonian fluid. The non-Newtonian fluid dynamics are important in connection with plastics manufacturing, performance of lubricants, application of paints, and movement of biological fluids.^{10,11} The power-law model is the well-known constitutive equation for non-Newtonian fluid and is widely used in engineering work. When the power-law exponent, n , is equal to 1, the fluid becomes Newtonian. If $n < 1$, the fluid is said to be "pseudoplastic" or "shear thinning," and if $n > 1$, the fluid is called "dilatant" or "shear thickening." Dimitrov¹² has reviewed the literature on the thinning and stability of some non-Newtonian liquid films, and Sathiyagal¹³ used a linear theory to predict the critical thickness of rupture for a power-law liquid film. In the present investigation, we illustrate the application of the power-law model in a thin liquid-film system and study its nonlinear rupture mechanism.

Considering a thin viscous liquid film on a horizontal plane as shown in Fig. 1. The local film thickness, h , is thin enough to neglect the gravity effect, but is affected by the van der Waals attraction. Assuming that the liquid is a power-law model, the fluid is defined as

$$\tau_{ij} = m |\dot{\gamma}|^{n-1} \cdot \dot{\gamma}_{ij}, \quad (1)$$

where τ_{ij} is the stress tensor, $\dot{\gamma}_{ij}$ is the rate-of-strain tensor, n is the power-law exponent, and m is the preexponential constant. By using the following scales—time [$\sim h_0(\rho h_0^n/m)^{-1/(n-2)}$], length ($\sim h_0$), velocity [$\sim (\rho h_0^n/m)^{1/(n-2)}$], pressure, and stress [$\sim \rho(\rho h_0^n/m)^{2/(n-2)}$], and with the quasisteady lubrication assumption, we can formulate the following leading order equations:

$$u_x + w_z = 0, \quad (2)$$

$$p_x + \phi_x = nu_z^{n-1} u_{zz}, \quad (3)$$

$$p_z = 0, \quad (4)$$

with the following boundary conditions:

$$\text{at } z=0,$$

$$u=w=0; \quad (5)$$

$$\text{at } z=h,$$

$$u_z=0, \quad (6)$$

$$p = -3Sh_{,xx}, \quad (7)$$

and the kinematic condition at the free surface is given by

$$h_t + uh_x = w \quad \text{at } z=h, \quad (8)$$

where u , w , and p are the dimensionless x -component velocity, the z -component velocity, and pressure, respectively. $\phi = Ah^{-3}$ is the van der Waals potential. The dimensionless parameters in the system (2)–(8) are

$$A = A'/6\pi h_0^3 D \quad \text{and} \quad S = \sigma/3h_0 D, \quad (9)$$

where $D = \rho(\rho h_0^n/m)^{2/(n-2)}$, A' is the dimensional Hamker constant, and σ is the surface tension.

We solve Eqs. (2)–(4), subject to boundary conditions (5)–(7), to find the solutions of velocity in the x and z component:

$$u = \frac{n}{n+1} \Phi^{1/n} [-(h-z)^{(n+1)/n} + h^{(n+1)/n}],$$

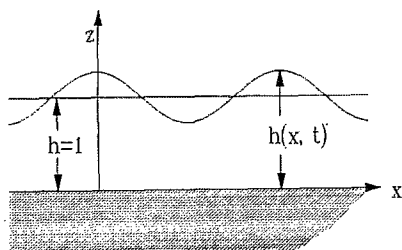


FIG. 1. The diagram of a thin liquid-film system.

$$\begin{aligned}
 w = & -\frac{n}{n+1} \Phi^{1/n} \left((h-z)^{(n+1)/n} h_{,x} + \frac{n+1}{n} h^{1/n} h_{,xz} \right) \\
 & - \frac{1}{n+1} \Phi^{(1-n)/n} \Phi_{,x} \left(\frac{n}{2n+1} (h-z)^{(2n+1)/n} \right. \\
 & \left. + h^{(n+1)/n} z \right) + \frac{n}{n+1} \Phi^{1/n} h^{(n+1)/n} h_{,x} \\
 & + \frac{n}{(n+1)(2n+1)} \Phi^{(1-n)/n} \Phi_{,x} h^{(2n+1)/n}, \quad (10)
 \end{aligned}$$

where

$$\Phi = 3Sh_{,xxx} + 3Ah^{-4}h_{,x}.$$

We then substitute solutions (10) into Eq. (8) to obtain the following evolution equation:

$$h_{,t} + \frac{n}{2n+1} \left[(3Sh_{,xxx} + 3Ah^{-4}h_{,x})^{1/n} h^{(2n+1)/n} \right]_{,x} = 0. \quad (11)$$

When $n=1$, this equation reduces to that derived by Williams and Davis.⁷

We solve nonlinear partial differential equation (13) numerically by using the finite-difference methods. Centered differences are employed in space while the midpoint rule is used for time and the Newton-Raphson method is

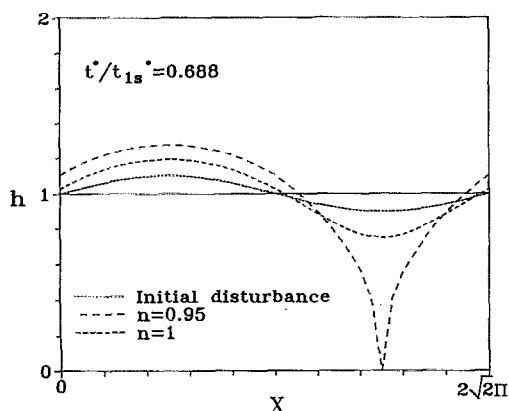


FIG. 2. The film profile at an initial time and at a later time for different values of n .

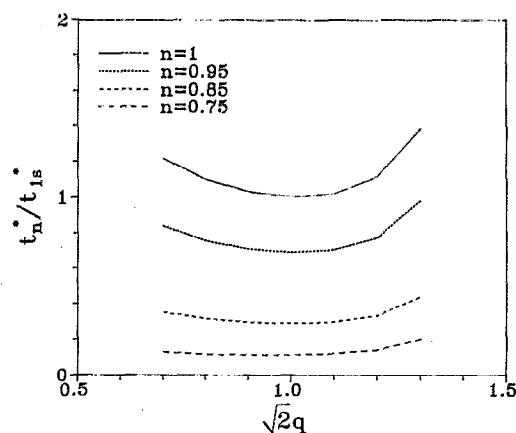


FIG. 3. The relative rupture time, t_n^*/t_{1s}^* , vs the wave number of disturbance, q , for different values of n .

introduced to solve the difference equations. The convergent tolerance of the Newton-Raphson iteration method is 10^{-6} .

Since the dimensionless time is defined in terms of the power-law exponent, n , the numerical results for the rupture with different values of n are difficult to compare directly. Therefore, we display the numerical results in the real time scale, t^* , to remove the mentioned difficulty. Figure 2 exhibits the film profile at an initial time and at later time for different values of n . We take the shortest rupture time for Newtonian fluid t_{1s}^* as the new reference time. This figure shows that when $t/t_{1s}^* = 0.688$ the liquid film of $n=0.95$ is broken but the one for an n value of 1 is not. It is because the smaller n value results in a smaller viscosity. This causes an increasingly unstable factor and makes the liquid film rupture more quickly. Figure 3 displays the relative rupture time, t_n^*/t_{1s}^* , versus the wave number of disturbance, q , for different values of n . It shows that the mode of the shortest rupture time will not change obviously when changing the value of n . The wave numbers of these most unstable modes are always near $1/\sqrt{2}$. From the above analysis and discussion we can finally conclude that reducing the power-law exponent, n , will strongly reduce the rupture time. However, the mode of the most unstable disturbance will not change.

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¹A. Vrij and J. Th. G. Overbeek, *J. Am. Chem. Soc.* **90**, 3074 (1968).

²B. U. Felderhof, *J. Chem. Phys.* **49**, 44 (1968).

³R. K. Jain, C. Maldarelli, and E. Ruckenstein, *AIChE Symp. Ser. Biorheol.* **74**, 120 (1978).

⁴P. M. Bisch and A. Sanfeld, *Bioelectrochem. Bioenerg.* **5**, 401 (1978).

⁵C. Maldarelli and R. K. Jain, *J. Colloid Interface Sci.* **90**, 263 (1982).

⁶E. Ruckenstein and R. K. Jain, *Chem. Soc. Faraday Trans. 2* **70**, 132 (1974).

⁷M. B. Williams and S. H. Davis, *J. Colloid Interface Sci.* **90**, 220 (1982).

- ⁸B. Reisfeld and S. G. Bankoff, *Phys. Fluids A* **2**, 2066 (1990).
- ⁹M. C. Wu and C. C. Hwang, *Int. Comm. Heat Mass Transfer* **18**, 705 (1991).
- ¹⁰C. Mathew Mate, Max R. Lorenz, and V. J. Novotny, *J. Chem. Phys.* **90**, 7550 (1989).
- ¹¹R. Byron Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids* (Wiley, New York, 1987).
- ¹²D. S. Dimitrov, *Prog. Surf. Sci.* **14**, 295 (1983).
- ¹³A. N. Sathyagal and G. Narsimhan, *Chem. Eng. Comm.* **111**, 161 (1992).