

On the Rupture Process of Thin Liquid Film

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Received August 10, 1992; accepted February 11, 1993

This present work aims at researching the rupture process of thin liquid film of viscous Newtonian fluid on a horizontal plane. First, we use a numerical method to solve the nonlinear evolution equation derived by Williams and Davis. The numerical results reveal that the rupture processes can be divided into two parts; one is the slow weak nonlinear process which almost dominates the whole rupture time, and the other is the rapid strong nonlinear process which just occupies a little rupture time. Second, we use the weak nonlinear theory to solve the nonlinear film equation, and modify this method to match the real process. Eventually, we can obtain the approximation analytical solutions which could predict the nonlinear film rupture time more quickly and correctly. © 1993 Academic Press, Inc.

1. INTRODUCTION

The research of rupture of thin liquid films has been applied not only in mechanical and chemical industry but also in biological engineering. Some industrial applications of thin film rupture are in foams and emulsions, floating thin coatings, oil recovery, vapor condensation on tube walls, and detergent action in the cleaning of oil films. Biological engineering application involves the small scale deformations of membranes, such as the onset of microvilli in normal and neoplastic cells, the deformation and rhythmic movement of red blood cells, adhesion, fusion, and attachment of cells and microorganisms, and phagocytosis. Recently this research is being applied to the basic study of aerospace technology.

If a liquid film is thin enough, rupture often occurs at a critical thickness h_c . Sheludko (1) addressed the fact that negative pressure thins the film thickness, but surface tension causes the opposite result. Ruckenstein and Jain (2) studied the rupture problem of a thin liquid film on a horizontal plate. They modeled the liquid as a Navier-Stokes continuum with extra body force due to van der Waals attraction, and solved it by using the linear stability theory. Williams and Davis (3) primarily derived a long-wave nonlinear evolution equation and solved it numerically. Sharma and

Ruckenstein (4) began with the Williams and Davis equation and sought to obtain a "quick" analytic solution. In the mean time, Burelbach *et al.* (5) studied numerically solutions of the Williams and Davis equation in detail. Reisfeld and Bankoff (6) and Wu and Hwang (7) analyzed the film rupture with viscosity variations. Their results showed that cooling (heating) from a wall would increase (decrease) the rupture time.

Although Sharma and Ruckenstein (4) have used the high harmonic method to obtain a quick analytic solution of the film rupture problem, their solution cannot match the real rupture processes. In this paper we examine the whole rupture process by a numerical method first, then improve the weak nonlinear theory to solve the nonlinear evolution equation in order to derive a quick and precise analytical solution of the Williams and Davis equation.

2. NUMERICAL ANALYSIS OF THE WILLIAM AND DAVIS EQUATION

Consider a thin viscous liquid film on a horizontal plane as shown in Fig. 1. Here h is the local film thickness. Repeating the methods of William and Davis (3), we have the long wave evolution equation:

$$\begin{aligned} h_{,T} + (h^{-1}h_{,X} + h^3h_{,XXX})_{,X} + \frac{A^2}{S} \left\{ \frac{2}{5} h^5 [Ah^{-4}h_{,X} \right. \\ \left. + Sh_{,XXX}]_{,T} + \frac{5}{8} h^4 h_{,T} (Ah^{-4}h_{,X} + Sh_{,XXX}) \right. \\ \left. + \frac{111}{280} h^7 (Ah^{-4}h_{,X} + Sh_{,XXX}) [Ah^{-4}h_{,X} + Sh_{,XXX}]_{,X} \right. \\ \left. + \frac{39}{40} h^6 h_{,X} (Ah^{-4}h_{,X} + Sh_{,XXX})^2 \right\}_{,X} = 0. \quad [1] \end{aligned}$$

In Eq. [1], X and T are rescaling dimensionless spatial and temporal variables, and A and S are parameters corresponding to the Van der Waals potential and the surface tensor, respectively. With the $A \approx 10^{-4}$, $S \approx 10^{-1}$ used by Williams and Davis (3), the order of A^2/S is much smaller

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than 1, thus Eq. [1] reduces to the Williams and Davis equation,

$$h_{,T} + (h^{-1}h_{,X} + h^3h_{,XXX})_{,X} = 0. \quad [2]$$

Since the equation is correct up to order about A^2/S , the purpose of this paper now focuses on solving Eq. [2] in a much deeper approach. Williams and Davis (3) used the normal method to get the linear rupture time in order to compare with the nonlinear rupture time. Later, Sharma and Ruckenstein (4) and Burelbach *et al.* (5) also introduced the linear rupture time as the reference time. Here we also use the normal mode method to examine the linear rupture time and verify it by numerical method. We substitute $h = 1 + H(X, T)$ into Eq. [2], yielding the linearized equation

$$H_{,T} + H_{,XX} + H_{,XXXX} = 0. \quad [3]$$

Substituting

$$H(X, T) = H_0 \exp(\omega T + iqX) \quad [4]$$

into [3] produces the characteristic equation

$$\omega = q^2 - q^4. \quad [5]$$

In Eqs. [4] and [5], H_0 is the initial disturbances amplitude, ω is the disturbance growth rate, and q denotes the wave number. From $d\omega/dq = 0$, the wave number of maximum growth rate q_M is

$$q_M = \frac{1}{\sqrt{2}}, \quad [6]$$

the maximum growth rate ω_M is

$$\omega_M = \frac{1}{4}, \quad [7]$$

and finally the shortest linear rupture time T_{LM} is

$$T_{LM} = -4 \ln(H_0). \quad [8]$$

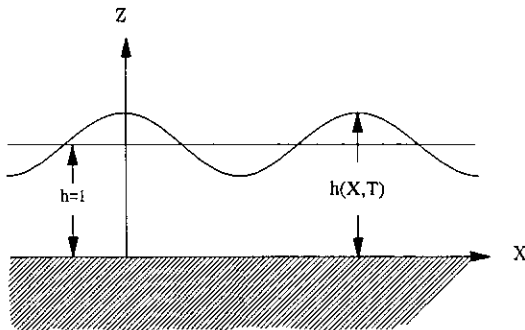


FIG. 1. The diagram of a thin liquid film system.

Equation [2] is a strongly nonlinear partial differential equation, and is solved by the Crank–Nicolson and Newton–Raphson iteration methods for a spatially periodic solution on the fixed interval $0 < X < 2\pi/q_M$. Burelbach *et al.* (5) showed that if the time interval ΔT is fixed at 0.01, spatial effects become unimportant when the spatial-mesh number $N > 20$. In this paper we choose $\Delta T = 0.001$, $N = 50$, $H_0 = 0.05 \sim 0.25$, and the convergent error limit of iteration method is 10^{-6} .

Figure 2 shows the film rupture processes of linear and nonlinear evolution equations with $H_0 = 0.1$. It reveals that linear rupture process is slow and smooth, but the nonlinear rupture process is quicker and steeper. In order to make the rupture process more clear, we choose the lowest interface height $H_{\min}$ and its velocity $-H'_{\min}$ of the numerical solution, to express the rupture processes. Figure 3 shows the variations of $-H'_{\min}$ and $H_{\min}$ with respect to time; Fig. 3a reveals that $-H'_{\min}$ will increase rapidly after $H_{\min}$ reaches the range of $0.6 \sim 0.7$. This means that after the weak nonlinear stage the strong nonlinear process will evolve with high acceleration. Figure 3b shows that the rupture processes can be divided into two parts, one the slow weak nonlinear process which almost dominates the whole rupture time, and the other the rapid strong nonlinear process which just occupies a small portion of rupture time.

3. WEAK NONLINEAR THEORY

The numerical results reveal that the weak nonlinear process dominates the whole rupture process, so we introduce the weak nonlinear method to match the real process, i.e., to expand the film thickness as the series

$$H = 1 + \epsilon H_1(X, T) + \epsilon^2 H_2(X, T) + \epsilon^3 H_3(X, T) + \dots, \quad [9]$$

where $\epsilon < 1$ is a small parameter which is related to the initial amplitude of perturbation.

After substituting Eq. [9] into Eq. [2] and solving them order by order, we have the first harmonic, the second harmonic, and the third harmonic solutions as follow:

$$H_1 = \sin q_M X e^{\omega_M T}, \quad [10]$$

$$H_2 = \frac{q_M^2(1 + 3q_M^2)}{2\omega_M - 4q_M^2(1 - 4q_M^2)} \cos 2q_M X e^{2\omega_M T}, \quad [11]$$

and

$$H_3 = \left[\frac{(1/4)q_M^2(2 + 39q_M^2)}{3\omega_M - q_M^2(1 - q_M^2)} \sin q_M X - \frac{3q_M^2(1 + 6q_M^2)}{3\omega_M - 9q_M^2(1 - 9q_M^2)} \sin 3q_M X \right] e^{3\omega_M T}. \quad [12]$$

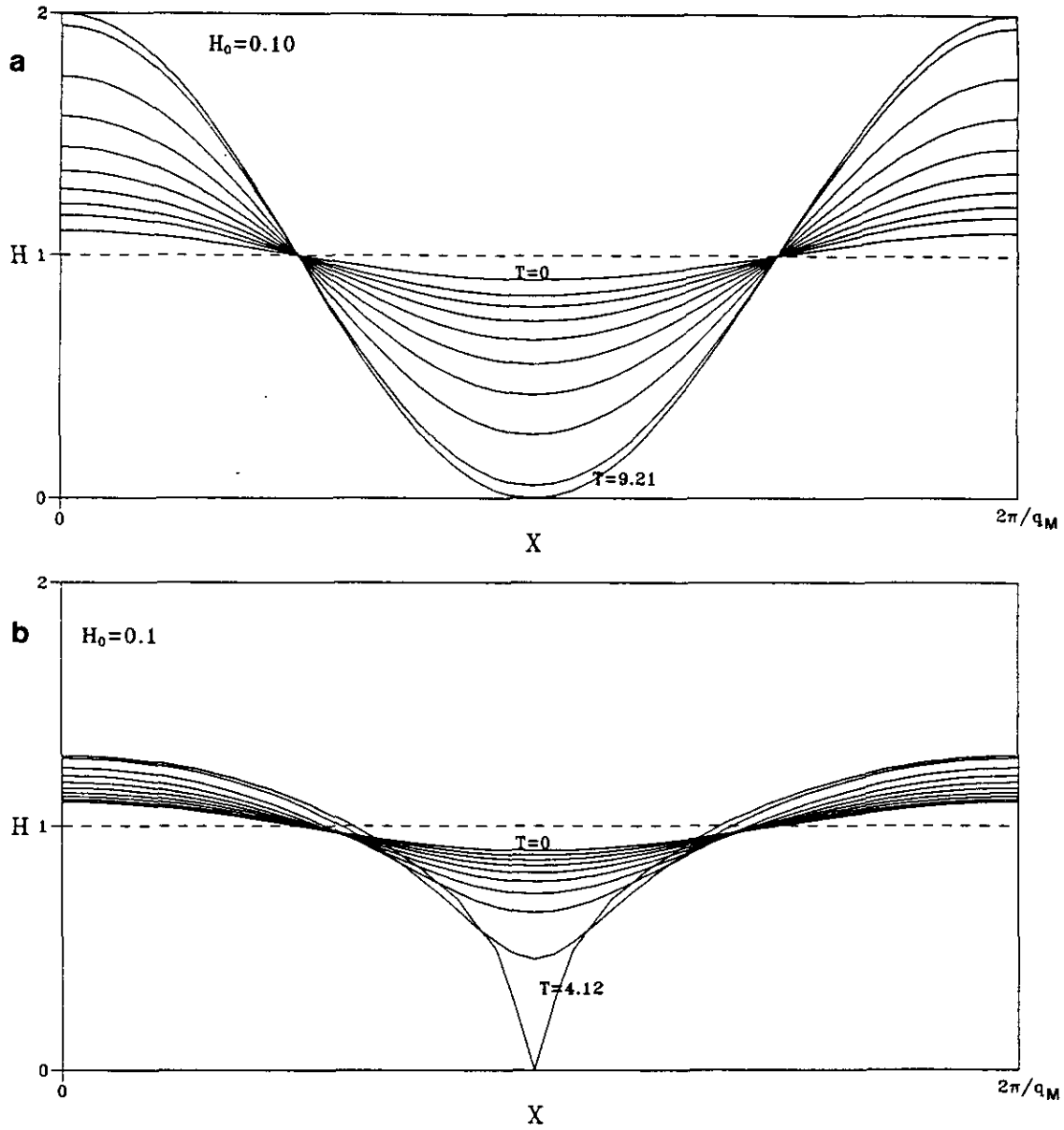


FIG. 2. (a) Linear rupture process, (b) nonlinear rupture process.

So the weak nonlinear solution of Eq. [2] can be expressed as

$$\begin{aligned}
 H = & 1 + \epsilon \sin q_M X e^{\omega_M T} \\
 & + \epsilon^2 \frac{q_M^2 (1 + 3q_M^2)}{2\omega_M - 4q_M^2 (1 - 4q_M^2)} \cos 2q_M X e^{2\omega_M T} \\
 & + \epsilon^3 \left[\frac{(1/4)q_M^2 (2 + 39q_M^2)}{3\omega_M - q_M^2 (1 - q_M^2)} \sin q_M X \right. \\
 & \left. - \frac{3q_M^2 (1 + 6q_M^2)}{3\omega_M - 9q_M^2 (1 - 9q_M^2)} \sin 3q_M X \right] e^{3\omega_M T}. \quad [13]
 \end{aligned}$$

The solution Eq. [13] is different from that proposed by Sharma and Ruckenstein (4), for they neglected the term $\sin q_M X$ in the third harmonic solution, Eq. [12]. Equation [13] indicates the thickness of the thin liquid film at different positions and time. In order to get the rupture time we locate the position of the lowest interface height $H_{\min}$ by letting

$$\frac{dH}{dX} = 0, \quad \frac{d^2 H}{dX^2} > 0. \quad [14]$$

The rupture will occur at the position $q_M X = \frac{3}{2}\pi$, then the

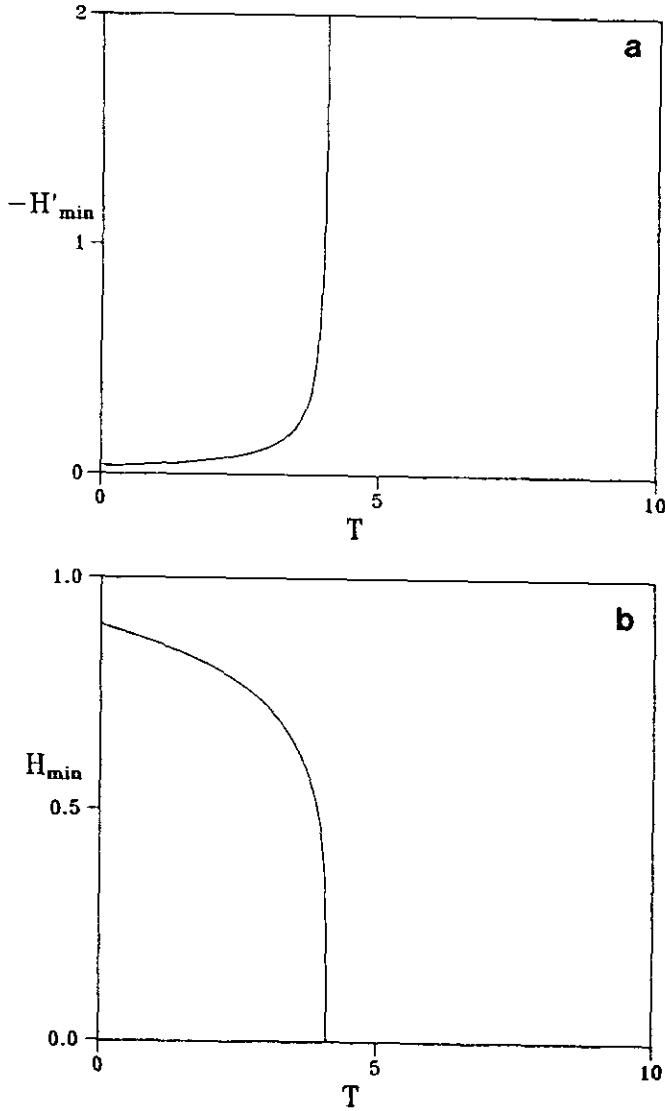


FIG. 3. (a) $-H'_{\min}$ varied with time, (b) $H_{\min}$ varied with time, numerical results with $H_0 = 0.1$.

variation of the lowest interface height $H_{\min}$ with time can be described by

$$H_{\min} = 1 - \epsilon e^{T/4} - \frac{1}{2} \epsilon^2 e^{T/2} - \frac{505}{88} \epsilon^3 e^{(3/4)T}. \quad [15]$$

Figure 4 shows the comparison among linear and nonlinear numerical results, and weak nonlinear analytic results of variations of $-H'_{\min}$ and $H_{\min}$ with time, where $H'_{\min}$ means the velocity of $H_{\min}$. In Fig. 4a, it is found that the weak nonlinear results cannot predict the overall process, but the numerical results show that the weak nonlinear process occupies most of the rupture time. Figure 4b shows that the correct time range of the weak nonlinear process is about $1 \leq H_{\min} \leq 0.6$. However, Sharma and Ruckenstein (4) took

$H_{\min}$ to reach zero to get the rupture time. Consequently, their results are different from the numerical result shown in Fig. 5. Since the weak nonlinear process occupies most of the rupture time, one can approach it by using Eq. [15] and taking $H_{\min}$ about equal to $0.6 \sim 0.7$. Figure 5 shows that, compared to the exact numerical solutions, the results of taking $H_{\min} = 0.6$ are better. This means that in the present case the rupture time can be obtained from the following empirical relation,

$$1 - \epsilon e^{T/4} - \frac{1}{2} \epsilon^2 e^{T/2} - \frac{505}{88} \epsilon^3 e^{(3/4)T} = 0.6, \quad [16]$$

where ϵ is relative to the initial disturbance amplitude H_0 ,

$$\epsilon \left(1 + \frac{1}{2} \epsilon^2 + \frac{505}{88} \epsilon^2 \right) = H_0. \quad [17]$$

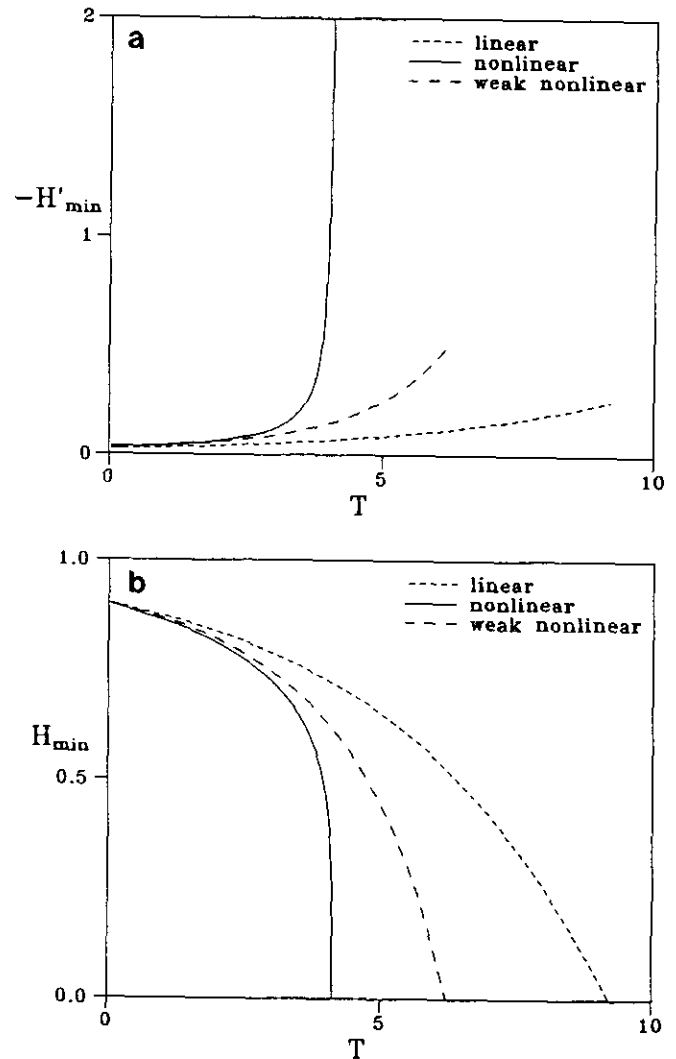


FIG. 4. (a) $-H'_{\min}$ varied with time, (b) $H_{\min}$ varied with time, comparison among the results of different methods with $H_0 = 0.1$.

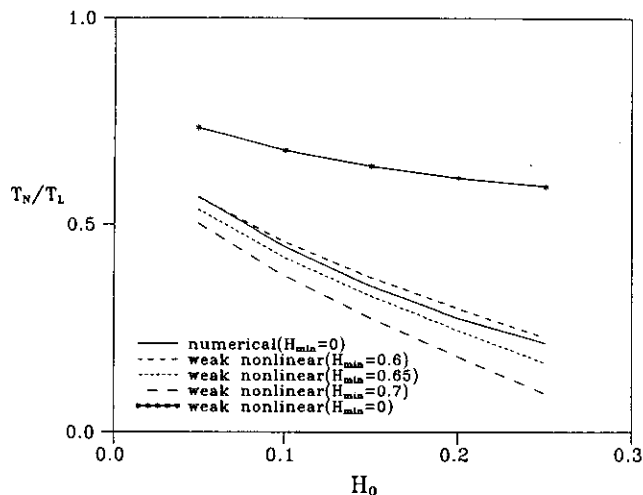


FIG. 5. The weak nonlinear rupture time with different values of $H_{\min}$, compared with numerical results.

Usually the numerical iteration method takes much computer time to solve the nonlinear rupture time. However, we can get the rupture time more quickly and correctly by Eq. [16]. Apparently, this weak nonlinear method can be considered as a paragon method for the study of film rupture problems in the future.

4. CONCLUSIONS

In the present work we study the rupture process of the liquid film by solving the Williams and Davis long-wave

equation. In the numerical results, we find that $-H'_{\min}$ increases rapidly after the $H_{\min}$ evolve in the range of about 0.6 ~ 0.7. This means that the rupture processes can be divided into two parts: one is the slowly weak nonlinear process which almost dominates the whole rupture time, and the other is the rapidly strong nonlinear process which just occupies a small portion of the rupture time. Then we modify the weak nonlinear method to find the nonlinear rupture time. The modified method improves the results of the original weak nonlinear method, and additionally this improved analytical solution can predict the nonlinear film rupture time quickly and correctly.

ACKNOWLEDGMENT

The authors acknowledge with appreciation the financial support (Grant NSC 81-F-SP-033-01) provided by the National Science Council of the Republic of China.

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