

Nonlinear Rupture Theory of a Thin Liquid Film on a Cylinder

JUN-LIANG CHEN AND CHI-CHUAN HWANG¹

Department of Mechanical Engineering, Chung Yuan University, Chung-Li, Taiwan 32023, Republic of China

Received January 31, 1996; accepted April 29, 1996

The dynamic rupture process of a thin liquid film on a cylinder is investigated numerically. First a nonlinear differential equation that describes the long-wave evolution of the interface shape is derived. After the evolution equation is solved and analyzed, the results reveal that a decrease in the cylinder radius will induce a stronger lateral capillary force and consequently will accelerate the rupture process. Moreover, the tendency of acceleration becomes more explicit in the case of large surface tension. © 1996

Academic Press, Inc.

Key Words: nonlinear rupture; thin liquid film; van der Waals potential.

I. INTRODUCTION

The hydrodynamic stability of thin liquid films has been widely studied allowing for many applications in chemical, mechanical, and biomedical engineering. A liquid layer is unstable when the layer is ultrathin (100–1000 Å). The instability is due to van der Waals potential and results in the rupture of the layer. The difference in chemical potential with respect to large phases of the same material of the film makes all the corresponding intensive thermodynamic properties change (1, 2).

Vrij (3) used a static stability analysis to find the critical thickness of a nondraining film affected by the van der Waals forces. Ruckenstein and Jain (4) studied the dynamic linear stability of the rupture problem of a thin liquid film on a horizontal plane with the effect of van der Waals potential. Williams and Davis (5) and Burelbach *et al.* (6) analyzed the long-wave nonlinear dynamic rupture process of a film on a plate and examined the local flow structure near the rupture point. Chen and Hwang (7) derived a system of evolution equations, which included the inertia effect neglected in Williams and Davis's equation, by the approximate Kármán–Polhausen integral theory. Furthermore, Hwang *et al.* (8) used the same integral method to derive an evolution equation system by considering the higher-order viscous dissipation effect that seldom being considered before.

There has been much research on the stability of a thin liquid film on the cylindrical surface (9–14). However, little has been reported on the effect of the van der Waals potential. Also, there has been some other research focused on the adhesion and spreading of liquids on a cylinder. They used the static theory to predict the critical condition of spreading. Gelfand and Lipowsky (15) studied a fluid system in contact with a uniformly curved substrate, such as a cylinder or a sphere. By using the interface model and the Landau theory, they found that the effects of curvature may be subsumed into an effective bulk field. Upton *et al.* (16) used the Landau theory to study the wetting phenomena on spherical and cylindrical substrates with the critical temperature effects. Brochard (17) discussed the spreading of liquids on thin cylinders by considering the effects of attractive van der Waals forces and found that for volatile liquids, the vapor would adhere to the cylinders; the equilibrium thickness of the film is given by a balance between the disjoining van der Waals pressure and the Laplace pressure. However, for nonvolatile liquids, a liquid drop cannot spread out over the cylinder if the spreading coefficient is smaller than a critical value. Quéré *et al.* (18) studied the spreading of liquids on a highly curved surface. They showed two different ways to prevent the Rayleigh instability from developing on fibers coated with a liquid film. If the liquid film is very small, long-range van der Waals forces with negative Hamaker constant can stabilize it. On the other hand, if the film is thick, one can prevent the unstable sheath from breaking up by holding the fiber vertically and thus saturating the stability.

In practice, there are many superslender cylinders, such as an optical fiber, with its radius being fabricated to less than 10 μm (17, 19–21). Additionally, if the thickness of a liquid coating film is thin enough to the degree around 10^3 Å then the van der Waals potential may affect obviously the stability of this film flow system. Nevertheless, to date there has been no report on the dynamic analysis of this fluid flow system. In this paper, first we derive a nonlinear evolution equation to study the axisymmetrical rupture phenomena of a thin liquid coating film on a vertical cylinder. Then both the methods of linear analysis and nonlinear numerical calcu-

¹ To whom correspondence should be addressed.

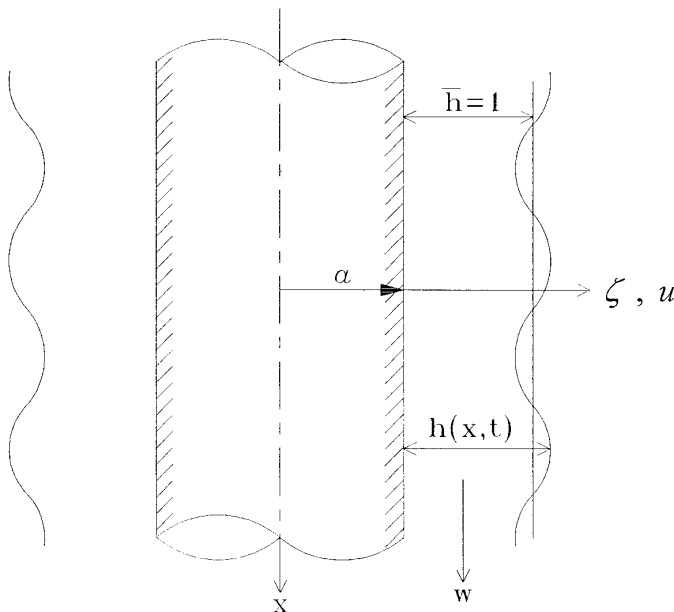


FIG. 1. The physical model of the system.

lation are applied to the evolution equation. Finally, we discuss the effects of the cylinder radius and the van der Waals potential on the rupture process.

II. FORMULATION OF NONLINEAR EVOLUTION EQUATION

A thin liquid film on a vertical cylinder is shown in Fig. 1. The liquid is a Newtonian viscous fluid having kinematic viscosity ν and constant density ρ . Besides, the local film thickness, h , is assumed to be thin enough to neglect the gravity effect, and is simply affected by the van der Waals potential. The perturbation of film flow is assumed to be axisymmetrical so that the azimuthal component of the flow is neglected. By scaling the spatial coordinate by the equilibrium thickness h_0 , time by h_0^2/ν , velocity by ν/h_0 , pressure by $\rho\nu^2/h_0^2$, respectively, one can formulate the dimensionless governing equations as follows:

$$\frac{1}{a + \zeta} [(a + \zeta)u]_\zeta + w_x = 0, \quad [1]$$

$$u_t + uu_\zeta + wu_x$$

$$= -p_\zeta + \frac{1}{a + \zeta} [(a + \zeta)u_\zeta]_\zeta - \frac{u}{(a + \zeta)^2} + u_{xx}, \quad [2]$$

$$w_t + uw_\zeta + ww_x$$

$$= -p_x - \phi_x + \frac{1}{a + \zeta} [(a + \zeta)w_\zeta]_\zeta + w_{xx}; \quad [3]$$

boundary conditions at $\zeta = 0$,

$$u = w = 0; \quad [4]$$

at $\zeta = h$

$$(u_x + w_\zeta)(1 - h_x^2) - 2h_x(u_\zeta + w_x) = 0, \quad [5]$$

$$-p + 2(1 + h_x^2)^{-1} [(u_\zeta + h_x^2 w_x) - (u_x + w_\zeta)h_x] \\ = 3S(1 + h_x^2)^{-3/2} [h_{xx} - \frac{1}{a + h} (1 + h_x^2)], \quad [6]$$

$$h_t + wh_x = u, \quad [7]$$

where u , w , p , and a are the dimensionless ζ -component velocity, x -component velocity, pressure, and cylinder radius. The r -coordinate is scaled as $r = a + \zeta$, where a is given by r_i/h_0 with r_i being denoted by the dimensional radius of the cylinder. Also S , the dimensionless surface tension, is defined as $S = h_0\sigma/(3\rho\nu^2)$, where σ is the dimensional surface tension. Equation [1] is the continuity equation, and Eqs. [2] and [3] are momentum equations in the ζ and x directions, respectively. Equations [4], [5], and [6] are the no-slip condition on the solid surface, the shear, and the normal stress balance condition at the free surface. Equation [7] is the kinematic boundary condition at the free surface. In the present study, the attractive van der Waals forces are represented by the potential function which depends on the film thickness as (22)

$$\phi = Ah^{-3},$$

where A , the dimensionless Hamaker, constant can be correlated to the dimensional Hamaker constant A' by $A = A'/(6\pi h_0\rho\nu^2)$. Note that for thicker films the presence of the electromagnetic retardation effect should be considered, and then the van der Waals potential will be modified as (23)

$$\phi = Ah^{-4}.$$

A long-wave approximation is used to obtain successive approximations to the solutions of Eqs. [2] and [3]. The solutions derived for each component of the velocity field will be substituted into the kinematic equation [7]. We formalized the long-wave theory by rescaling the variables with a small parameter ϵ . We defined new scaled variables as follows:

$$x = \epsilon^{-1}X, \quad \zeta = Z, \quad t = \epsilon^{-4}\tau, \quad \phi = \epsilon^2\Phi, \\ u = \epsilon^4U, \quad w = \epsilon^3W, \quad p = \epsilon^2P, \quad h = H. \quad [8]$$

The scaled variables and derivatives with respect to the upper case variables are all of unit order. We also assume the order of the parameters as

$$A = O(\epsilon^2), \quad S = O(1), \quad \text{and} \quad a = O(\epsilon^{-1})$$

to retain those effects at the lowest order.

Approximate solutions of this new system of equations can be obtained by introducing the following regular perturbation expansion for U , V , and P into the governing equations

$$(U, W, P) = (U_0, W_0, P_0) + \epsilon^2(U_1, W_1, P_1) + \cdots \quad [9]$$

Then the governing equations and the corresponding boundary conditions concerned with the zeroth-order problem can be derived as follows:

$$U_{0Z} + W_{0X} = 0, \quad [10]$$

$$W_{0ZZ} = P_{0X} + \Phi_X, \quad [11]$$

$$P_{0Z} = 0; \quad [12]$$

at $Z = 0$,

$$U_0 = W_0 = 0; \quad [13]$$

at $Z = H$,

$$W_{0Z} = 0, \quad [14]$$

$$P_0 = -3SH_{XX} - 3Sa^{-2}(a - H), \quad [15]$$

$$H_\tau + W_0H_X = U_0. \quad [16]$$

The first term in the right-hand side of Eq. [15] is the streamwise component of surface tension, and the second term is the lateral capillary force. It is clear that the streamwise surface tension is affected by the magnitude of S . Meanwhile, the lateral capillary force is influenced strongly by both the magnitude of the cylinder radius and S . We solved Eqs. [10]–[12] simultaneously with the solutions being subject to the boundary conditions [13]–[15] and finally obtained the solutions of velocity in the x component as follows:

$$W_0 = -\frac{3}{2}(SH_{XXX} + SH_Xa^{-2} + AH^{-4}H_X)(Z^2 + HZ). \quad [17]$$

By substituting W_0 into Eq. [16] and rewriting the equation in terms of the original variables, the evolution equation can be obtained as

$$h_t + (Sh^3h_{xxx} + Sa^{-2}h^3h_x + Ah^{-1}h_x)_x = 0. \quad [18]$$

As $a \rightarrow \infty$, Eq. [9] reduces to

$$h_t + (Sh^3h_{xxx} + Ah^{-1}h_x)_x = 0, \quad [19]$$

which is identical to that derived by Williams and Davis (5) for a film on a flat plane. If the film thickness is thicker than the present system, the effect of gravity should be taken into consideration and the primary flow of the thin coating liquid film will be present. Meanwhile, the effect of the van der Waals potential is too small to exist, and the order of magnitude of the cylinder radius should be assumed to be $O(1)$. Under the consideration mentioned above, a new evolution equation analogous to Eq. [18] is obtained, and it is equivalent to Eq. [A2] of Lin and Liu (10).

III. RESULTS AND DISCUSSION

It is instructive to analyze the linear stability of the steady-state solution $h = 1$. Introducing the deviation $h' = h - 1$ into Eq. [18] and expressing the solution of the equation as

$$h' = H_0 \exp(\omega t + ikx) + \text{c.c.},$$

where c.c. denotes the complex conjugate, one can find the following characteristic equation:

$$\omega = Ak^2 + Sa^{-2}k^2 - Sk^4. \quad [20]$$

Here ω is the growth rate and k is the wavenumber. The cut-off wavenumber or the wavenumber of neutral state k_c is found at $\omega = 0$ and can be expressed as

$$k_c = \left[\frac{A}{S} + \frac{1}{a^2} \right]^{1/2}. \quad [21]$$

Meanwhile, the wavenumber of maximum growth rate k_m is given at $d\omega/dk = 0$ and can be shown as

$$k_m = \left[\frac{A}{2S} + \frac{1}{2a^2} \right]^{1/2}, \quad [22]$$

which is in good agreement with the results predicted by the linear theory presented by Williams and Davis (5) when $a \rightarrow \infty$. Taking $1 + h'(x, t) = 0$ for at least one value of x , we can find linear rupture time, T_L , given by

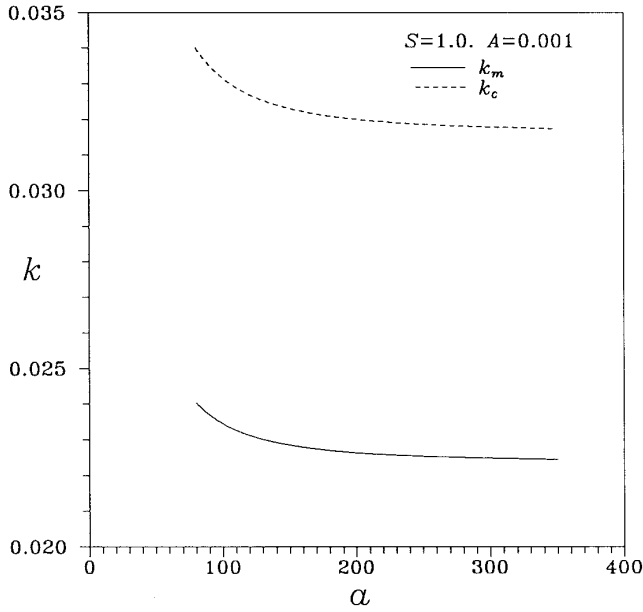


FIG. 2. The curves for k_m and k_c versus a .

$$T_L = -\frac{1}{\omega_m} \ln(H_0), \quad [23]$$

where H_0 is the initial disturbance.

It is obvious from Eqs. [21]–[22] that both the cut-off wavenumber and the maximum growth rate decrease as the cylinder radius increases. That is, the domain of the unstable mode shrinks and the film becomes unstable as the effect of lateral capillary force becomes weaker, as demonstrated in Fig. 2. Figure 3 displays the linear evolution of the lowest

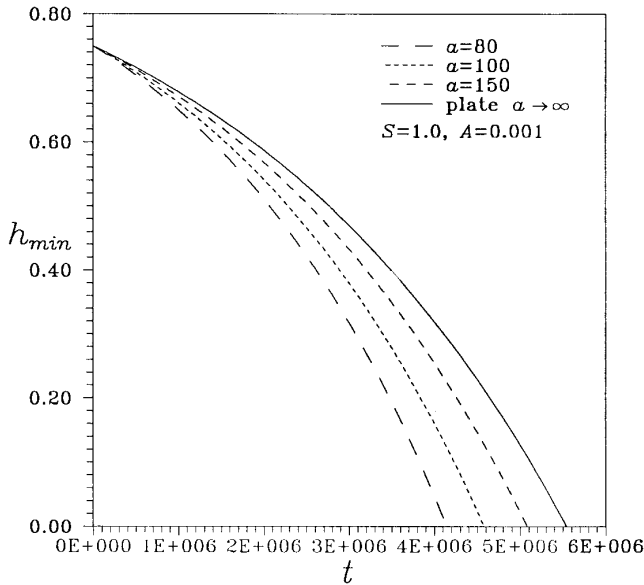


FIG. 3. $h_{\min}$ versus time under various a values for linear theory.

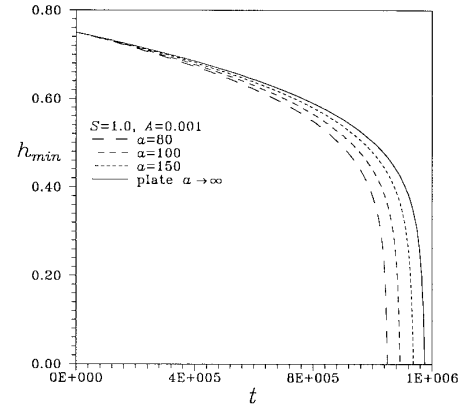


FIG. 4. $h_{\min}$ versus time under various values of a for nonlinear theory.

point of the film thickness $h_{\min}$. The initial disturbance is 0.25 in this figure. The results show that there is no particular acceleration phenomenon during the whole linear rupture process. Additionally, the rupture time predicted by the linear analysis increases when the radius of cylinder is enlarged. This phenomenon is thought to result from the fact that increasing the cylinder radius will decrease the effect of lateral capillary force. Considering the terms containing the surface tension in the characteristic equation, we find that the surface tension has dual effects on the stability characteristics. This is because the surface tension in the streamwise component has stabilizing effects, but the lateral capillary force has a destabilizing effect. The stabilizing mechanism of the streamwise surface tension had been studied in past research (2–6). Here, the linear destabilizing mechanism of the lateral capillary force can be described as follows: the bottom of the perturbed surface waves has a larger capillary pressure than that of peak of the waves. This would induce the capillary pressure force to push fluid from the bottom to the peak, and consequently, increase the amplitude of the wave.

The nonlinear rupture process of system is revealed by solving evolution equations numerically. Equations are discretized by using the finite-difference method. Center differences are employed in space, while the midpoint rule is used for time. The calculation domain is fixed on the interval $0 \leq x \leq 2\pi/k_m$. The periodic boundary conditions are considered in this calculation. The initial condition is

$$h(x, 0) = 1.0 + H_0 \cos k_m x, \quad 0 \leq x \leq 2\pi/k_m \quad [24]$$

where H_0 is the initial disturbance. The Newton–Raphson iteration method is introduced to calculate difference equations. The convergent tolerance is 10^{-5} and the initial disturbance H_0 is 0.25.

The lowest point of film thickness $h_{\min}$ varies with time for nonlinear theory, as shown in Fig. 4. In this figure, one

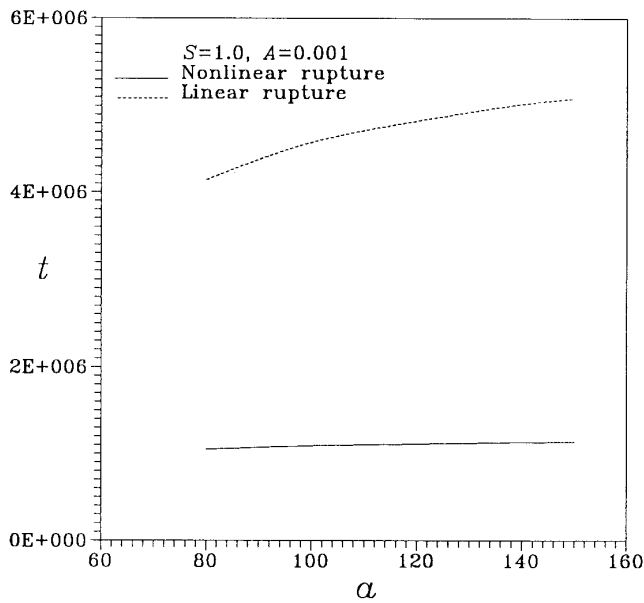


FIG. 5. Linear and nonlinear rupture time versus α .

can find that the rupture process is accelerated as $h_{\min}$ approaches zero. The rupture process begins to proceed very fast during the range $h_{\min} \leq 0.65 \sim 0.5$ (6). This acceleration process occurs earlier as the cylinder radius is smaller. Figure 5 shows the linear and nonlinear rupture times versus α . It is found that the linear rupture time is larger than that of the nonlinear one. The difference between the linear and nonlinear rupture times becomes obvious as α increases. This result is the same as that of a film on flat plane: the effects of the nonlinear terms, which accompany α , enlarge the acceleration of the film rupture process (3–6). Figure 5 also shows that both the linear and the nonlinear rupture times decrease as the radius of the cylinder shrinks, but the tendency of the linear theory is stronger than that of the nonlinear one. This is due to the fact that the streamwise surface tension becomes important in the nonlinear ranges.

The variation of the nonlinear rupture time following the change of A is shown in Fig. 6. The rupture time decreases as the effect of the van der Waals potential increases. This phenomenon is similar to those results shown in previous studies (5–8). Additionally, the difference between the rupture time of the planar film and that of the cylindrical film decreases as the magnitude of A increases. When A varies from 0.001 to 0.003, the variation of $|(T_{NC} - T_{NP})/T_{NP}|$ is from 11 to 1%, where T_{NC} is the nonlinear rupture time of the cylindrical film and T_{NP} is that of the planar film. The lateral capillary force effects does not dominate the rupture process in the case of a strong effect of the van der Waals potential. In Fig. 7, one can see that the surface tension effects delay the rupture process. Moreover, the difference between the rupture times of the planar film and that of the

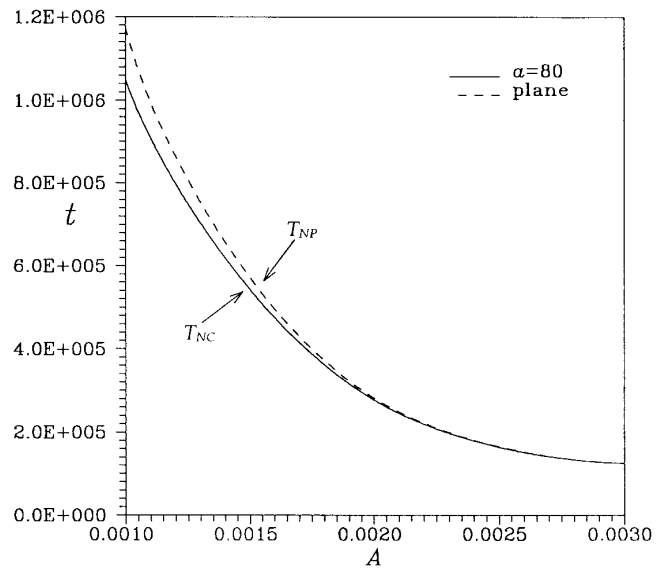


FIG. 6. Nonlinear rupture time versus A ; $S = 1$.

cylindrical film increases as S value increases. When S varies from 1 to 3, the variation of $|(T_{NC} - T_{NP})/T_{NP}|$ is from 11 to 25%. The strong lateral capillary force induces an acceleration effect on the rupture process as the magnitude of S becomes large. Therefore, one can conclude that there are two ways to enlarge the effect of lateral capillary force; one is shrinking the cylinder radius, and the other is increasing the S value.

IV. CONCLUSIONS

The effect of the lateral capillary force of the cylinder on the stability of a cylindrical film has been studied by taking

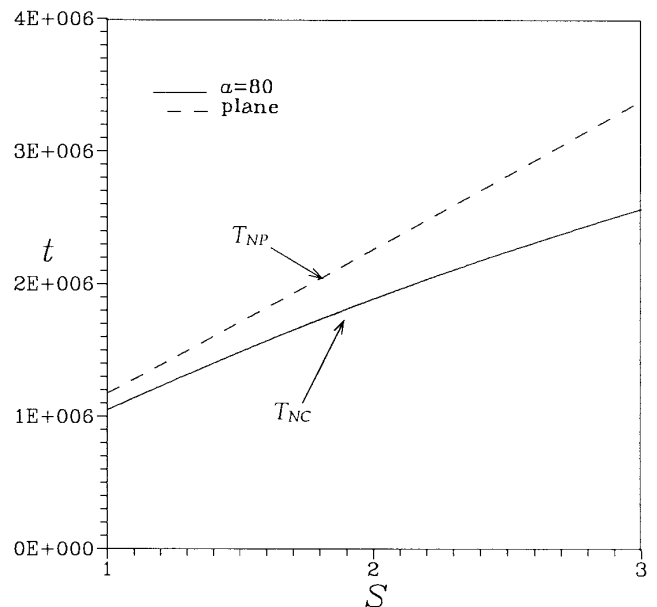


FIG. 7. Nonlinear rupture time versus S ; $A = 0.001$.

into account the influence of the van der Waals attraction and the surface tension. It is shown that the nonlinear dynamics of such a system can be described by the simple model of an evolution system for the evolution of the film thickness. A preliminary linear analysis of the model allows us to obtain the critical wavenumber of the system, which depends on the van der Waals potential, surface tension, and cylinder radius. The linear analysis results show that an increase in the radius of cylinder makes the system more stable. Moreover, the numerical method has been used to study the nonlinear rupture process. Under the same conditions as those for linear analysis, the nonlinear rupture time is shorter than the linear one. However, enlarging the lateral capillary force will decelerate the rupture process in both the linear and nonlinear cases.

ACKNOWLEDGMENT

The authors acknowledge with appreciation the financial support (Grant No. NSC 85-2212-E033-008) provided by the National Science Council of the Republic of China.

REFERENCES

1. Sheludko, A., *Adv. Colloid Interface Sci.* **1**, 391 (1967).
2. Overbeek, J. T. G., *J. Phys. Chem.* **64**, 1178 (1960).
3. Vrij, A., *Discuss. Faraday Soc.* **42**, 23 (1966).
4. Ruckenstein, E., and Jain, R. K., *J. Chem. Soc. Faraday Trans. II* **70**, 132 (1974).
5. Williams, M. B., and Davis, S. H., *J. Colloid Interface Sci.* **90**, 220 (1982).
6. Burelbach, J. P., Bankoff, S. G., and Davis, S. H., *J. Fluid Mech.* **195**, 463 (1988).
7. Chen, J.-L., and Hwang, C.-C., *J. Colloid Interface Sci.* **167**, 214 (1994).
8. Hwang, C.-C., Chen, J.-L., and Shen, L.-F., *Phys. Rev. E* (revised).
9. Hasegawa, E., and Nakaya, C., *J. Phys. Soc. Jpn.* **29**, 1634 (1970).
10. Lin, S. P., and Liu, W. C., *AIChE J.* **21**, 775 (1975).
11. Krantz, W. B., and Zollars, R. L., *AIChE J.* **22**, 930 (1976).
12. Tougou, H., *J. Phys. Soc. Jpn.* **43**, 318 (1977).
13. Solorio, F. J., and Sen, M., *J. Fluid Mech.* **183**, 365 (1987).
14. Cheng, M., and Chang, H.-C., *Chem. Eng. Commun.* **118**, 327 (1992).
15. Frenkel, A. L., *Phys. Fluids A* **5**, 2342 (1993).
16. Gelfand, M. P., and Lipowsky, R., *Phys. Rev. B* **36**, 8725 (1987).
17. Upton, P. J., Indekeu, J. O., and Yeomans, J. M., *Phys. Rev. B* **40**, 666 (1989).
18. Borchard, F., *J. Chem. Phys.* **84**, 4664 (1986).
19. Quéré, D., Di Meglio, J.-M., and Brochard-Wyart, F., *Science* **249**, 1256 (1990).
20. Yeh, C., "Handbook of Fiber Optics." Academic Press, San Diego, 1990.
21. Baker, D. G., "Fiber Optic Design and Applications." Prentice-Hall, Virginia, 1985.
22. Reisfeld, B., and Bankoff, S. G., *J. Fluid Mech.* **236**, 167 (1992).
23. Lin, S. P., and Brenner, H., *J. Colloid Interface Sci.* **89**, 226 (1982).