



INVERSE DYNAMICS OF A TOGGLE MECHANISM

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Abstract—The inverse dynamics problem aimed at determining the driving forces that produce a specific motion is presented. It is necessary to know the velocities and accelerations to be able to estimate the inertia forces which provide the basis to calculate the required actuating forces. A constrained multibody technique is employed to calculate the position, velocity and acceleration of the toggle mechanism. For the multibody dynamic analysis, Hamilton's principle and Lagrange multiplier method are applied to formulate the equations of motion. By using geometric relationships the differential-algebraic equations can be reduced to pure differential equations form and Runge-Kutta numerical method is used to perform numerical simulations. The results of kinematic and dynamic analysis are presented and compared for the four-point type and the five-point toggle mechanisms. © 1997 Elsevier Science Ltd. All rights reserved.

NOTATION

A_B, A_C	the acceleration vectors of sliders B and C , respectively
f	the offset between link 1 and slider 4
h	the distance between sliders B and C
h_B, h_C	the position vectors of sliders B and C , respectively
I_i	the mass moments of inertia of link i
m_2, m_3, m_5	the mass of link 2, 3 and 5, respectively
m_B, m_C	the mass of sliders B and C , respectively
P_B, P_C	the applied forces acting on the sliders B and C , respectively
r_1, r_2, r_4	the three-side lengths of link 2
r_3, r_5	the lengths of links 3 and 5, respectively
T_i	the total kinetic energy
V_B, V_C	the velocity vectors of sliders B and C , respectively
x_B, x_C	the positions of sliders B and C in x axis, respectively
Φ_B, Φ_C	the constraint vectors of sliders B and C , respectively
θ	the generalized coordinates vector
$\theta_1, \theta_2, \theta_3$	the angular positions of links 2, 3 and 5, respectively
$\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$	the angular velocities of links 2, 3 and 5, respectively
$\ddot{\theta}_1, \ddot{\theta}_2, \ddot{\theta}_3$	the angular accelerations of links 2, 3 and 5, respectively
ϕ	the angle between links 3 and 4

INTRODUCTION

Multibody dynamics has grown in the past decade to be an important tool in the design and simulation of complex mechanical systems. Multibody systems consist of collections of rigid bodies that are

constrained to move relatively to one another by kinematic connections between bodies. Toggle mechanism is an application of slider-crank mechanism, it is widely used in various types of mechanisms. This mechanism has many applications where it is necessary to overcome a large resistance force with a small driving force, e.g. clutches, rock crushers, truck tailgates, vacuum circuit breakers, pneumatic riveters, punching machines, forging machines, injection molding machines, etc.

Another important feature of the toggle mechanism is its ability to produce high values of force at the slider with relatively low torque input. Mostofi [1] investigated particular attention to: (i) the collapse of the toggle in the presence of Coulomb friction; and (ii) the effect of a varying periodic force on the mechanism with damped elastic coupling. Beer and Johnston [2] applied the principle of virtual work to analyze the toggle vise. Burton [3] considered the toggle mechanism with hinge friction. In most cases, mechanisms made up of more than four links may be broken down into simple basic linkages and solved in a straightforward manner. The traditional toggle mechanism is a combination of two basic linkages, a four-bar mechanism and a slider-crank mechanism. The kinematic analysis of the entire toggle mechanism is performed by considering the two basic linkages separately (Wilson and Sadler [4] and Martin [5]). Blajer [6] addressed two main issues: how to model the dynamics of a system with non-ideal constraints, and how to solve the resulting governing equations. Blejwas [7] applied the Lagrange multiplier method to solve a slider-crank mechanism. Chang and Nikravesh [8] applied the constraint

violation stabilization method to the dynamic analysis of mechanical systems.

In the Lagrange multiplier method for dynamic analysis, the motion of each link is represented in terms of its own rigid-body generalized coordinates. The links of toggle mechanism are treated as coupled to each other except for the application of interaction forces which enforce the constraint conditions between the links. Rather than eliminating the multipliers and obtaining coupled system coordinates, the Lagrange multipliers are solved in time as part of a numerical technique. The multipliers are applied in turn to the individual links and the simulation proceeds to the next point in time using numerical integration.

In this paper, a new toggle mechanism, shown in Fig. 1, is designed by the combination of two slider-crank mechanisms. The cycloidal motion is considered as the trajectory of slider *B*. The corresponding position, velocity and acceleration in the given stroke (230 mm) of slider *B* are to be studied by an analytical method. Equations specifying the positions of the various links in the toggle mechanism can be written using trigonometry, and the velocity and acceleration equations can be derived by differentiation of the position equations. The dynamic motion is derived by the Hamilton's principle and Lagrange multiplier method as differential-algebraic equations. The inverse dynamics is applied for deriving the input force of slider *C*. The coordinate partitioning theorem is applied here to provide a theoretical reduction of differential-algebraic equations to differential equation form. Therefore, we obtain the valid existence and uniqueness results for differential-algebraic equations of motion for the constrained systems.

Kinematic analysis

The physical model is shown in Fig. 1, where links 1, 2, 3 and slider *B* comprise a slider-crank mechanism with an offset *f*. If the external force

exerts on the slider *C*, then links 5, 2 and 3 are driven and the output force at slider *B* rises. The height between the two horizontal guides where sliders *B* and *C* move along is denoted as *h*. It is convenient to call Fig. 1 a five-point toggle mechanism for $r_2 \neq 0$, and a four-point toggle mechanism for $r_2 = 0$, $r_4 = r_1$ and $\phi = 0$.

Degree of freedom

The degree of freedom (DOF) of the toggle mechanism can be found from the Gruebler's equation:

$$\text{DOF} = 3(L - 1) - 2J \quad (1)$$

and taking $L = 6$ and $J = 7$, we obtain the DOF of the system is one.

This toggle mechanism consists of an assembly of links, each of whose motion is constrained by the contact with other links. Links 1, 2, 3 and slider 4 comprise a slider-crank mechanism and links 1, 2, 5 and slider 6 also comprise another one. From the geometry of Fig. 1, we can find relational equations in the *y* direction as

$$r_1 \sin \theta_1 = f + r_3 \sin(2\pi - \theta_2) \quad (2)$$

$$r_5 \sin \theta_5 + r_4 \sin(\theta_1 + \phi) = h + f. \quad (3)$$

It can also be presented as two independent holonomic constraint equations as follows

$$\Phi(\theta) = \begin{bmatrix} r_1 \sin \theta_1 + r_3 \sin \theta_2 - f \\ r_5 \sin \theta_5 + r_4 \sin(\theta_1 + \phi) - h - f \end{bmatrix} = 0 \quad (4)$$

where θ is the vector of three generalized coordinates, θ_5 , θ_2 and θ_1 . Thus, it can be ensured that the DOF of this system is one. A crucial ingredient in

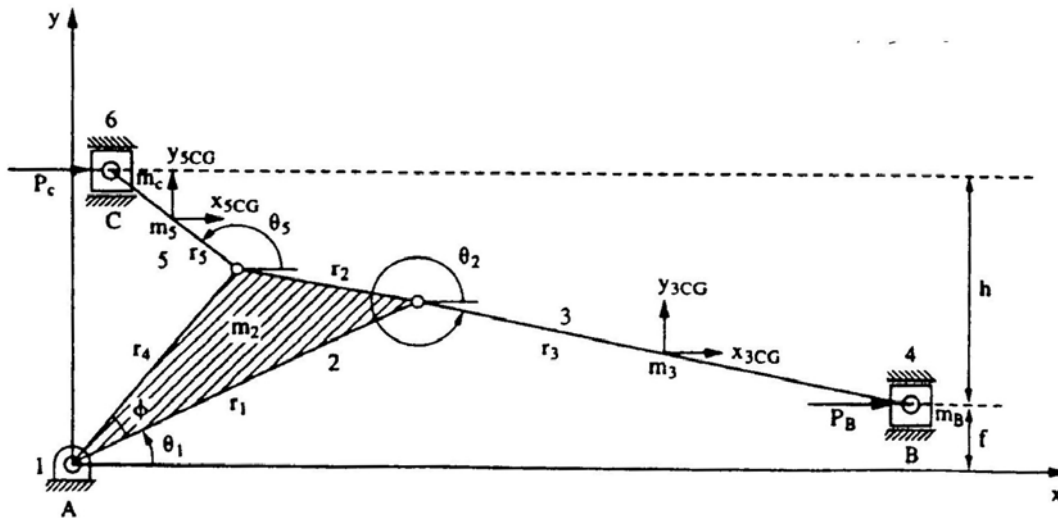


Fig. 1. Physical model of the five-point type toggle mechanism.

formulating well-posed dynamics is to assure that the constraint Jacobian

$$\Phi_\theta = \left[\frac{\partial \Phi_s(\theta)}{\partial \theta_i} \right] = \begin{bmatrix} 0 & r_3 \cos \theta_2 & r_1 \cos \theta_1 \\ r_5 \cos \theta_5 & 0 & r_4 \cos(\theta_1 + \phi) \end{bmatrix} \quad (5)$$

has full row rank (i.e. the constraints are independent).

The kinematic velocity and acceleration equations are obtained by taking the first and second time derivatives of eqn (4), respectively, as

$$\Phi_\theta \dot{\theta} = 0 \quad (6)$$

$$\Phi_\theta \ddot{\theta} = -(\Phi_\theta \dot{\theta})_\theta \dot{\theta} \equiv \gamma \quad (7)$$

where

$$\Phi_\theta \dot{\theta} = \begin{bmatrix} r_3 \cos \theta_2 \dot{\theta}_2 + r_1 \cos \theta_1 \dot{\theta}_1 \\ r_5 \cos \theta_5 \dot{\theta}_5 + r_4 \cos(\theta_1 + \phi) \dot{\theta}_1 \end{bmatrix}$$

$$\Phi_\theta \ddot{\theta} = \begin{bmatrix} r_3 \sin \theta_2 \dot{\theta}_2^2 + r_1 \sin \theta_1 \dot{\theta}_1^2 \\ r_5 \sin \theta_5 \dot{\theta}_5^2 + r_4 \sin(\theta_1 + \phi) \dot{\theta}_1^2 \end{bmatrix}$$

The first row of eqns (6) and (7) may be solved theoretically for the angular velocity and acceleration of the connecting rod, link 2, as

$$\dot{\theta}_2 = -\frac{r_1 \cos \theta_1}{r_3 \cos \theta_2} \dot{\theta}_1 \quad (8)$$

$$\ddot{\theta}_2 = \frac{\dot{\theta}_2}{\dot{\theta}_1} \dot{\theta}_1 + \frac{r_1 \dot{\theta}_1^2 \sin \theta_1 + r_3 \dot{\theta}_2^2 \sin \theta_2}{r_3 \cos \theta_2} \quad (9)$$

Similarly, the second row of eqns (6) and (7) may be solved theoretically for the angular velocity and acceleration of another connecting rod, link 5, as

$$\dot{\theta}_5 = -\frac{r_4 \cos(\theta_1 + \phi)}{r_5 \cos \theta_5} \dot{\theta}_1, \quad (10)$$

$$\ddot{\theta}_5 = \frac{\dot{\theta}_5}{\dot{\theta}_1} \dot{\theta}_1 + \frac{r_5 \dot{\theta}_5^2 \sin \theta_5 + r_4 \dot{\theta}_1^2 \sin(\theta_1 + \phi)}{r_5 \cos \theta_5} \quad (11)$$

where ϕ can be found by using the cosine theorem as

$$\phi = \cos^{-1} \left(\frac{r_1^2 + r_4^2 - r_2^2}{2r_1 r_4} \right) \quad (12)$$

The position vector of B can be expressed in the exponential form:

$$\mathbf{h}_B = r_1 e^{i\theta_1} + r_3 e^{i\theta_2} \quad (13)$$

Using eqn (2) and expanding eqn (13) in terms of trigonometry, we obtain

$$x_B = r_1 \cos \theta_1 + [r_3^2 - (f - r_1 \sin \theta_1)^2]^{1/2} \quad (14)$$

$$\theta_1 = \sin^{-1} \left[\frac{x_B^2 + r_1^2 + f^2 - r_3^2}{2r_1 \sqrt{x_B^2 + f^2}} \right] - \theta_1' \quad (15)$$

where x_B is the real part of $\mathbf{h}_B$ and

$$\theta_1' = \sin^{-1} \left[\frac{x_B}{\sqrt{x_B^2 + f^2}} \right]$$

By substituting eqn (15) into eqns (2) and (3), respectively, we can obtain

$$\theta_2 = 2\pi - \sin^{-1} \left[\frac{r_1 \sin \theta_1 - f}{r_3} \right] \quad (16)$$

$$\theta_5 = \sin^{-1} \left[\frac{h + f - r_4 \sin(\theta_1 + \phi)}{r_5} \right] \quad (17)$$

Then, the position vector of C can be found as follows:

$$\mathbf{h}_C = r_4 e^{i(\theta_1 + \phi)} + r_5 e^{i\theta_5} \quad (18)$$

and the real part of $\mathbf{h}_C$ is

$$x_C = r_4 \cos(\theta_1 + \phi) + r_5 \cos \theta_5 \quad (19)$$

If the position x_B of slider B is given, then θ_1 and θ_5 can be obtained from eqns (15) and (17), respectively. Sequentially, the position of slider C is obtained by substituting θ_1 and θ_5 into eqn (19).

For the analysis of velocity, we differentiate eqn (13) with respect to time and obtain the expression for the velocity of slider B as

$$\mathbf{V}_B = ir_1 \dot{\theta}_1 e^{i\theta_1} + ir_3 \dot{\theta}_2 e^{i\theta_2} \quad (20)$$

Substituting eqn (8) into eqn (20), we can obtain

$$\dot{\theta}_1 = \frac{V_B^x}{r_1 [\cos \theta_1 \tan \theta_2 - \sin \theta_1]} \quad (21)$$

where V_B^x is the real part of the velocity vector at slider B . Substituting eqns (15), (17) and (21) into eqn (10)

gives the value of θ_s , then the velocity V_C^x of slider C in the x direction is

$$\begin{aligned} V_C^x &= Re(V_C) \\ &= Re(ir_4\dot{\theta}_1 e^{i(\theta_1 + \phi)} + ir_5\dot{\theta}_5 e^{i\theta_5}) \\ &= -r_4\dot{\theta}_1 \sin(\theta_1 + \phi) - r_5\dot{\theta}_5 \sin \theta_5. \end{aligned} \quad (22)$$

The linear acceleration of slider B is obtained by taking twice differentiation of eqn (13) with respect to time as

$$\mathbf{A}_B = (-\ddot{\theta}_1^2 + i\ddot{\theta}_1)r_1 e^{i\theta_1} + (-\ddot{\theta}_2^2 + i\ddot{\theta}_2)r_3 e^{i\theta_2}. \quad (23)$$

Substituting eqn (9) into eqn (23), we have

$$\begin{aligned} \ddot{\theta}_1 &= \frac{-\dot{\theta}_1}{r_1\dot{\theta}_1 \sin \theta_1 + r_3\dot{\theta}_2 \sin \theta_2} \\ &\times \{A_B^x + r_1\dot{\theta}_1^2 \cos \theta_1 + r_3\dot{\theta}_2^2 \cos \theta_2 \\ &+ [r_1\dot{\theta}_1^2 \sin \theta_1 + r_3\dot{\theta}_2^2 \sin \theta_2]\tan \theta_2\} \end{aligned} \quad (24)$$

where A_B^x is the real part of acceleration vector at slider B . Substitution of $\ddot{\theta}_1$ into eqn (11) gives the value of $\ddot{\theta}_5$, then the acceleration A_C^x of slider C in the x direction is obtained as

$$\begin{aligned} A_C^x &= Re(A_C) = Re[(-\ddot{\theta}_1^2 + i\ddot{\theta}_1)r_4 e^{i(\theta_1 + \phi)} \\ &+ (-\ddot{\theta}_5^2 + i\ddot{\theta}_5)r_5 e^{i\theta_5}] \\ &= -r_4[\dot{\theta}_1^2 \cos(\theta_1 + \phi) + \ddot{\theta}_1 \sin(\theta_1 + \phi)] \\ &- r_5(\dot{\theta}_5^2 \cos \theta_5 + \ddot{\theta}_5 \sin \theta_5). \end{aligned} \quad (25)$$

Dynamic analysis

In this toggle mechanism, it is difficult to get the direct solutions. Therefore we apply the multibody dynamics method to the system.

First, we use energy method to find out the dynamic motion equation. The total kinetic energy of the system is

$$T_1 = T_2 + T_3 + T_5 + T_B + T_C.$$

Let T_2 , T_3 , T_5 , T_B and T_C represent the kinetic energies of links 2, 3, 5 and sliders B and C , respectively,

$$T_2 = \frac{1}{2}(I_{2A}\dot{\theta}_1^2) \quad (26)$$

$$T_3 = \frac{1}{2}(I_{3CG}\dot{\theta}_2^2 + m_3\dot{x}_{3CG}^2 + m_3\dot{y}_{3CG}^2) \quad (27)$$

$$T_5 = \frac{1}{2}(I_{5CG}\dot{\theta}_5^2 + m_5\dot{x}_{5CG}^2 + m_5\dot{y}_{5CG}^2) \quad (28)$$

$$T_B = \frac{1}{2}m_B\dot{x}_B^2 \quad (29)$$

$$T_C = \frac{1}{2}m_C\dot{x}_C^2 \quad (30)$$

where

$$I_{2A} = \frac{1}{2}m_2\left(\frac{r_1r_4}{r_2}\sin \phi\right)^2$$

$$I_{3CG} = \frac{1}{12}m_3r_3^2$$

$$I_{5CG} = \frac{1}{12}m_5r_5^2$$

$$\dot{x}_{3CG} = -r_1\dot{\theta}_1 \sin \theta_1 - \frac{r_3}{2}\dot{\theta}_2 \sin \theta_2$$

$$\dot{y}_{3CG} = -\frac{r_3}{2}\dot{\theta}_2 \cos \theta_2$$

$$\dot{x}_{5CG} = -r_4\dot{\theta}_1 \sin(\theta_1 + \phi) - \frac{r_5}{2}\dot{\theta}_5 \sin \theta_5$$

$$\dot{y}_{5CG} = -\frac{r_5}{2}\dot{\theta}_5 \cos \theta_5$$

$$\dot{x}_B = -r_1\dot{\theta}_1 \sin \theta_1 - r_3\dot{\theta}_2 \sin \theta_2$$

$$\dot{x}_C = -r_4\dot{\theta}_1 \sin(\theta_1 + \phi) - r_5\dot{\theta}_5 \sin \theta_5$$

can be found from the geometry of Fig. 1 and ϕ can be found from eqn (12).

The virtual work, denoted by δW^A , of the applied forces, P_B and P_C , acting on the sliders B and C with the virtual displacements δx_B and δx_C , respectively, is obtained as

$$\delta W^A = P_B\delta x_B + P_C\delta x_C. \quad (31)$$

Rewriting the eqn (31) in terms of the generalized coordinates θ ,

$$\delta W^A = -\delta\theta^T Q^A \quad (32)$$

where

$$Q^A = \begin{bmatrix} P_C r_5 \sin \theta_5 \\ P_B r_3 \sin \theta_2 \\ P_B r_1 \sin \theta_1 + P_C r_4 \sin(\theta_1 + \phi) \end{bmatrix}.$$

The generalized constraint reaction forces can be obtained in terms of Lagrange multipliers as $Q^C = -\Phi_\theta^T \lambda$, where Φ_θ can be found from eqn (5). Thus, the virtual work due to all constraint reaction forces is

$$\delta W^C = -\delta\theta^T Q^C. \quad (33)$$

The most general form of Hamilton's principle is

$$B\ddot{\theta}_2 + E\ddot{\theta}_1 + R\dot{\theta}_2^2 + S\dot{\theta}_1^2 + \lambda_1 r_3 \cos \theta_2 = P_B r_3 \sin \theta_2 \quad (39)$$

$$0 = \int_{t_1}^{t_2} [\delta T + \delta W^A + \delta W^C] dt$$

$$= \int_{t_1}^{t_2} \delta \theta^T \left[\frac{\partial T}{\partial \theta} - \frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}} - Q^A - Q^C \right] dt + \frac{\partial T}{\partial \dot{\theta}} \delta \theta \Big|_{t_1}^{t_2} \quad (34)$$

The varied path coincides with the true path at the two end points t_1 and t_2 . It follows that $\delta \theta(t_1) = \delta \theta(t_2) = 0$. The variational equations of eqn (34) must hold for all $\delta \theta$. Thus the equation of motion for the toggle mechanism is

$$M\ddot{\theta} + N(\theta, \dot{\theta}) + \Phi_\theta^T \lambda = Q^A \quad (35)$$

where the elements of the matrices M and N are detailed in Appendix A.

In addition to these equations, the constraint conditions including the position, velocity and acceleration equations in eqns (4), (6) and (7) must hold. Equation (35) and (7) may be written in matrix form as

$$\begin{bmatrix} M & \Phi_\theta^T \\ \Phi_\theta & 0 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \lambda \end{bmatrix} = \begin{bmatrix} Q^A - N(\theta, \dot{\theta}) \\ \gamma \end{bmatrix} \quad (36)$$

This is a system of differential-algebraic equations. Since no derivative of the Lagrange multiplier λ appears, the algebraic constraint equations of eqn (4) must be satisfied. Initial conditions for system motion are given as

$$\begin{aligned} \theta(0) &= \theta^0 \\ \dot{\theta}(0) &= \dot{\theta}^0 \end{aligned} \quad (37)$$

where θ^0 and $\dot{\theta}^0$ are consistent with constraints, that is, θ^0 satisfies eqn (4) and $\dot{\theta}^0$ satisfies eqn (6).

INVERSE DYNAMIC EQUATIONS

The main role in the dynamic analysis is the external force acting at slider C . The inverse dynamic equation is derived from the forward dynamic analysis. By using the inverse equation, the force applied at slider C is obtained while a certain type of motion at slider B is specified.

Equation (35) can be rewritten in three parts:

$$A\ddot{\theta}_5 + D\ddot{\theta}_1 + P\dot{\theta}_5^2 + Q\dot{\theta}_1^2 + \lambda_2 r_5 \cos \theta_5 = P_C r_5 \sin \theta_5 \quad (38)$$

$$\begin{aligned} D\ddot{\theta}_5 + E\ddot{\theta}_2 + C\ddot{\theta}_1 + T\dot{\theta}_5^2 + U\dot{\theta}_2^2 + V\dot{\theta}_1^2 \\ + \lambda_1 r_1 \cos \theta_1 + \lambda_2 r_4 \cos(\theta_1 + \phi) \\ = P_B r_1 \sin \theta_1 + P_C r_4 \sin(\theta_1 + \phi). \end{aligned} \quad (40)$$

From eqns (38) and (39), we have

$$\lambda_2 = \frac{1}{r_5 \cos \theta_5} [P_C r_5 \sin \theta_5 - A\ddot{\theta}_5 - D\ddot{\theta}_1 - P\dot{\theta}_5^2 - Q\dot{\theta}_1^2] \quad (41)$$

$$\lambda_1 = \frac{1}{r_3 \cos \theta_2} [P_B r_3 \sin \theta_2 - B\ddot{\theta}_2 - E\ddot{\theta}_1 - R\dot{\theta}_2^2 - S\dot{\theta}_1^2]. \quad (42)$$

By substituting eqns (41) and (42) into eqn (40), we obtain

$$\begin{aligned} P_C &= \frac{1}{r_4 [\cos(\theta_1 + \phi) \tan \theta_5 - \sin(\theta_1 + \phi)]} \\ &\times \{ P_B r_1 (\sin \theta_1 - \cos \theta_1 \tan \theta_2) + [\psi_1 A - D] \ddot{\theta}_5 \\ &+ [\psi_2 B - E] \ddot{\theta}_2 + [\psi_1 D + \psi_2 E - C] \ddot{\theta}_1 \\ &+ [\psi_1 P - T] \dot{\theta}_5^2 + [\psi_2 R - U] \dot{\theta}_2^2 \\ &+ [\psi_1 Q + \psi_2 S - V] \dot{\theta}_1^2 \} \end{aligned} \quad (43)$$

where

$$\psi_1 = \frac{r_4 \cos(\theta_1 + \phi)}{r_5 \cos \theta_5} \quad \psi_2 = \frac{r_1 \cos \theta_1}{r_3 \cos \theta_2}$$

and the elements (e.g. $A, B, C, \dots$) are described in Appendix A.

The required input force can be calculated by eqn (43) to achieve the desired motion of slider B . P_B is considered as the friction force, i.e. $P_B = -\mu m_B g \operatorname{sgn}(V_B)$, where μ is the coefficient of friction and g is the gravitational acceleration. The signum function is defined as follows:

$$\operatorname{sgn}(V_B) = \begin{cases} -1 & V_B < 0 \\ 0 & V_B = 0 \\ 1 & V_B > 0. \end{cases}$$

REDUCED SYSTEM OF DIFFERENTIAL EQUATIONS OF MOTION

The differential-algebraic equations of motion derived above are summarized in matrix form as eqn (36), and constraints are of the form in eqn (4). Implicit methods have been employed to solve

systems of equations of this form. However, it is generally concluded that the combined system of differential-algebraic equations is inherently stiff and their direct solution is computationally expensive (Gear, [9]; Wehage and Haug, [10]).

It is defined a partitioning of $\theta = [\mathbf{u}^T, \mathbf{v}^T]^T$ such that Φ_u is the submatrix of Φ_θ whose columns correspond to elements $\mathbf{u}$ of θ and Φ_v is the submatrix of Φ_θ whose columns correspond to elements $\mathbf{v}$ of θ . Thus $\mathbf{v}$ may be treated as a vector of independent variables and $\mathbf{u}$ as a vector of dependent variables. Note that this is just a computational implementation of the coordinate partitioning theorem and Φ_u must be nonsingular. The elements of the vectors $\mathbf{u}$, $\mathbf{v}$ and matrices Φ_u , Φ_v are detailed in the Appendix B.

The equations of motion of eqn (36) may be reordered and partitioned, according to the decomposition of θ into $\mathbf{u}$ and $\mathbf{v}$, as

$$\mathbf{M}^{uu}\ddot{\mathbf{u}} + \mathbf{M}^{uv}\ddot{\mathbf{v}} + \Phi_u^T\lambda = (\mathbf{Q}^u - \mathbf{N}^u) \quad (44a)$$

$$\mathbf{M}^{vu}\ddot{\mathbf{u}} + \mathbf{M}^{vv}\ddot{\mathbf{v}} + \Phi_v^T\lambda = (\mathbf{Q}^v - \mathbf{N}^v) \quad (44b)$$

$$\Phi_u\ddot{\mathbf{u}} + \Phi_v\ddot{\mathbf{v}} = \lambda \quad (44c)$$

where the mass matrices in the first two equations are submatrices of $\mathbf{M}$ and the vector functions on the right are portions of the vector $\mathbf{Q}$ and $\mathbf{N}$. Similarly, the velocity equation of eqn (6) can be written in partitioned form as

$$\Phi_u\dot{\mathbf{u}} + \Phi_v\dot{\mathbf{v}} = 0. \quad (45)$$

Since the coefficient matrix Φ_u of $\dot{\mathbf{u}}$ in eqn (45) is nonsingular, the dependent velocity $\dot{\mathbf{u}}$ may be written as

$$\dot{\mathbf{u}} = -\Phi_u^{-1}\Phi_v\dot{\mathbf{v}} \quad (46)$$

which is a function only of $\mathbf{v}$ and $\dot{\mathbf{v}}$. Similarly, the third of eqn (44) may be solved for $\ddot{\mathbf{u}}$, to obtain

$$\ddot{\mathbf{u}} = \Phi_u^{-1}[\gamma - \Phi_v\ddot{\mathbf{v}}] \quad (47)$$

which, after using eqn (46), may be written in terms of only $\mathbf{v}$, $\dot{\mathbf{v}}$ and $\ddot{\mathbf{v}}$.

Since the coefficient matrix Φ_u^T of λ in the first of eqns (44) is nonsingular, this equation may be solved for λ to obtain

$$\lambda = (\Phi_u^{-1})^T[\mathbf{Q}^u - \mathbf{N}^u - \mathbf{M}^{uv}\ddot{\mathbf{v}} - \mathbf{M}^{uu}\ddot{\mathbf{u}}] \quad (48)$$

Substituting this result and eqn (47) into eqn (44b) yields

$$\hat{\mathbf{M}}^v(\mathbf{v}, \dot{\mathbf{v}}, t)\ddot{\mathbf{v}} = \hat{\mathbf{Q}}^v(\mathbf{v}, \dot{\mathbf{v}}, t) \quad (49)$$

where

$$\begin{aligned} \hat{\mathbf{M}}^v &= \mathbf{M}^{vv} - \mathbf{M}^{vu}\Phi_u^{-1}\Phi_v - \Phi_v^T(\Phi_u^{-1})^T \\ &\quad \times [\mathbf{M}^{uv} - \mathbf{M}^{uu}\Phi_u^{-1}\Phi_v] \\ \hat{\mathbf{Q}}^v &= \mathbf{Q}^v - \mathbf{N}^v - \mathbf{M}^{vu}\Phi_u^{-1}\gamma - \Phi_v^T(\Phi_u^{-1})^T \\ &\quad \times [\mathbf{Q}^u - \mathbf{N}^u - \mathbf{M}^{uu}\Phi_u^{-1}\gamma]. \end{aligned}$$

The result is a set of differential equations in only the independent generalized coordinates $\mathbf{v}$ that are consistent with position and velocity constraints that act on the system. All terms in eqn (6) are determined as functions of $\mathbf{v}$ and $\dot{\mathbf{v}}$ through the constraint equations of eqn (4) and eqn (46). Solution of this system yields the complete system state, including Lagrange multipliers if constraint reaction forces are desired. By using the Runge-Kutta method, eqn (49) with initial conditions shown in eqn (44) determines $\mathbf{v}$ and $\dot{\mathbf{v}}$. Equation (4) determines $\mathbf{u}$, eqn (46) determines $\dot{\mathbf{u}}$, eqn (47) determines $\ddot{\mathbf{u}}$ and eqn (48) determines λ . Substituting these results into eqns (14), (18), (19), (21), (22) and (24), the positions, velocities and accelerations of sliders B and C are presented.

SIMULATION RESULTS AND DISCUSSIONS

Choosing the elements of toggle mechanism with the following dimensions: $r_1 = 145$ mm, $r_2 = 70$ mm, $r_3 = 190$ mm, $r_4 = 100$ mm, $r_5 = 55$ mm, $f = 25$ mm and $h = 80$ mm for the five-point type. For the case of four-point toggle mechanism, we have $r_1 = r_4 = 145$ mm, $r_2 = 0$ and $r_5 = 94.5$ mm and keep the other parameters be the same as those of the five-point toggle mechanisms. The mass specifications of the elements of toggle mechanism are as follows: $m_2 = 25$ kg, $m_3 = 30$ kg, $m_5 = 2.5$ kg, $m_B = 80$ kg and $m_C = 5$ kg. Note that in the four-point type, the mass moments of inertia of link 2 becomes $I_{2A} = \frac{1}{2}m_2r_1^2$.

Since sliders B and C move along horizontal guides, the velocity and acceleration in the y direction should be zero. The following results obtained are checked with $Im(\mathbf{V}_C) = Im(\mathbf{A}_C) = 0$ and $Im(\mathbf{V}_B) = Im(\mathbf{A}_B) = 0$.

In Fig. 1, the toggle position occurs when link 2 and 3 are collinear. The stroke of slider B is considered as 230 mm from its beginning position to the toggle positions. Two seconds are required to complete one stroke in such configuration. Since the system has one degree of freedom, the desired position at point B will correspond to one and only one output position at point C . Figure 2 shows the one-to-one position relationship between sliders B and C . At slider C , the stroke of four-point type, 276.143 mm (from $x_C = -139.328$ mm to $x_C = 136.815$ mm), is greater than, the stroke of the five-point type, 199.334 mm (from $x_C = -113.904$ mm to $x_C = 85.430$ mm). If we define

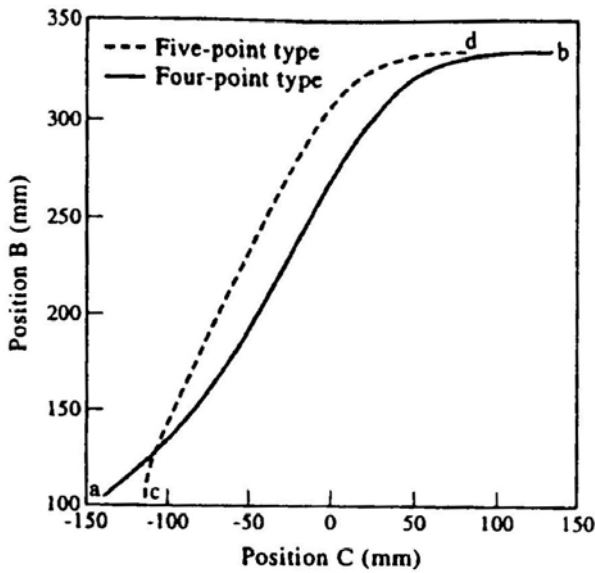


Fig. 2. One-to-one position relationship between B and C : (—) four-point type; (---) five-point type of toggle mechanisms; where $a = (-139.328, 104.066)$; $b = (136.815, 334.066)$; $c = (-113.904, 104.066)$; $d = (85.430, 334.066)$.

the stroke ratio as the ratio of the desired output stroke at slider B to the required input stroke at slider C , the stroke ratio, $230/276.143 = 0.833$, of the four-point type is less than $230/199.334 = 1.154$ of the five-point type. It means that in the same performance of the stroke at slider B , the four-point type needs longer stroke of slider C . Thus, from the view point of saving machine base, the five-point type is often applied in particle applications than the four-point type.

The graph in Fig. 3 shows the applied force at slider C where the friction coefficient μ is assumed as 0 and 0.1, respectively. It is seen that the input force at slider C of the five-point type is greater than the

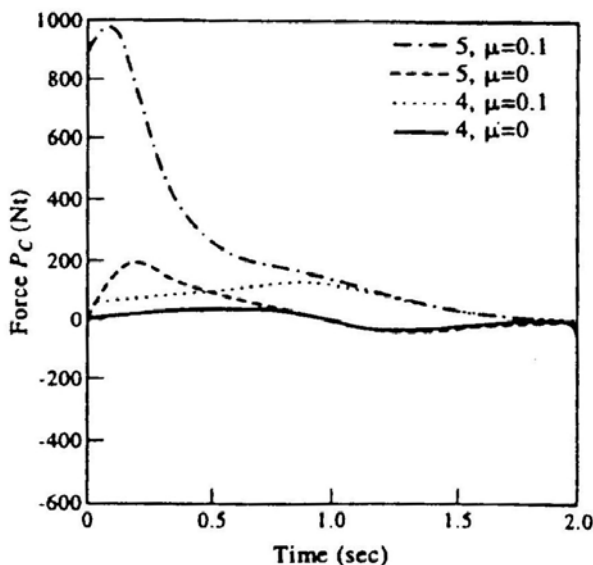


Fig. 3. The external force at slider C for the cycloidal motion: (—) four-point type with $\mu = 0$; (···) four-point type with $\mu = 0.1$; (---) five-point type with $\mu = 0$; (-·-) five-point type with $\mu = 0.1$.

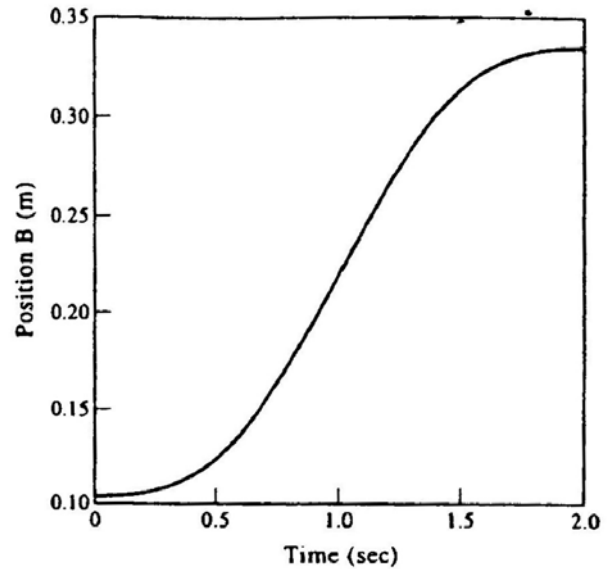


Fig. 4. Positions of slider B for the cycloidal motion: (—) four-point type with $\mu = 0$; (···) four-point type with $\mu = 0.1$; (---) five-point type with $\mu = 0$; (-·-) five-point type with $\mu = 0.1$.

four-point type. It is obvious that if a certain type of motion is given as desired motion, there is a unique applied force by using the inverse dynamic equations. On the other hand, if the same input force is applied to the direct dynamic equations, the same motions should be obtained. The position, velocity and acceleration of slider B during the time of a stroke are shown in Figs 4–6, respectively. It is seen that the curves are almost the same for both types of toggle mechanisms. When near the toggle positions (time = 1.7–2 s), the velocities and accelerations of five-point type (with $\mu = 0.1$) from dynamic results have some violation to the results of kinematic analysis.

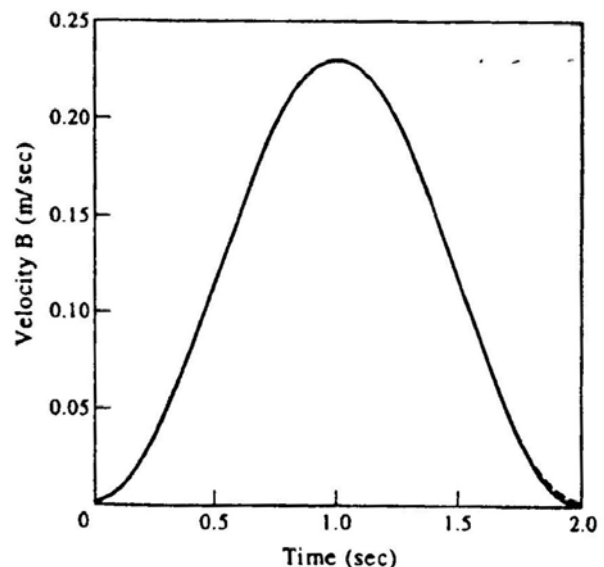


Fig. 5. Velocities of slider B for the cycloidal motion: (—) four-point type with $\mu = 0$; (···) four-point type with $\mu = 0.1$; (---) five-point type with $\mu = 0$; (-·-) five-point type with $\mu = 0.1$.

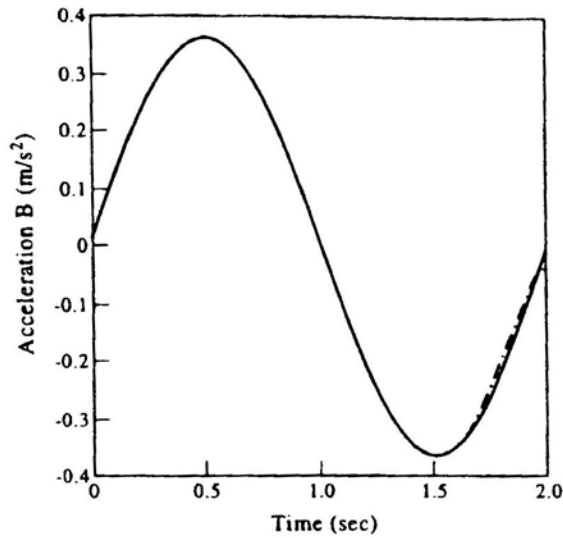


Fig. 6. Accelerations of slider B for the cycloidal motion: (—) four-point type with $\mu=0$; (···) four-point type with $\mu=0.1$; (---) five-point type with $\mu=0$; (- - -) five-point type with $\mu=0.1$.

CONCLUSIONS

This paper presents the kinematics, forward and inverse dynamics of the four-point type and the five-point type toggle mechanism. In the kinematic analysis, geometric relationships were used to calculate the position, velocity and acceleration at slider B of the desired cycloidal motions with respect to time. By using the Hamilton's principle and the Lagrange multiplier method, the dynamic motion of the mechanism was presented.

The one-to-one position relationship between B and C holds for the system with one degree of freedom. For the same stroke at slider B , the corresponding stroke at slider C of the four-point type is greater than that of the five-point type. However, the stroke ratio of the four-point type is less than that of the five-point one.

Inverse dynamics method provides a simple way to generate the required force profile. The results of using forward dynamics from the force profile are compared to that of the kinematic analysis. Although most of the results from dynamic analysis coincide with the kinematic ones, there are still some differences near the toggle position, especially in the five-point type. To improve this phenomenon, some feedback and tracking control techniques could be applied to this system, (Yao and Cheng, [11]; Kwon and Book, [12]).

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APPENDIX A

The elements of matrices M and N are given by

$$M = \begin{bmatrix} A & 0 & D \\ 0 & B & E \\ D & E & C \end{bmatrix}$$

$$N = \begin{bmatrix} P & 0 & Q \\ 0 & R & S \\ T & U & V \end{bmatrix} \begin{bmatrix} \theta_1^2 \\ \theta_2^2 \\ \theta_3^2 \end{bmatrix}$$

where

$$A = -\frac{1}{2} \left\{ \left(\frac{2}{3} m_s + 2m_c \sin^2 \theta_s \right) r_1^2 \right\}$$

$$B = -\frac{1}{2} \left\{ \left(\frac{2}{3} m_3 + 2m_b \sin^2 \theta_2 \right) r_1^2 \right\}$$

$$C = -\frac{1}{2} \left\{ \frac{m_2 r_1^2 r_4^2}{r_2^2} \sin^2 \phi + 2(m_1 + m_b) r_1^2 \sin^2 \theta_1 + 2(m_s + m_c) r_4^2 \sin^2(\theta_1 + \phi) \right\}$$

$$D = -\frac{1}{2} \{ (m_s + 2m_c) r_4 r_5 \sin(\theta_1 + \phi) \sin \theta_s \}$$

$$E = -\frac{1}{2} \{ (m_3 + 2m_b) r_1 r_3 \sin \theta_1 \sin \theta_2 \}$$

$$P = -\frac{1}{2} \{ 2m_c r_1^2 \sin \theta_s \cos \theta_s \}$$

$$Q = -\frac{1}{2} \{(m_5 + 2m_C)r_4r_5 \cos(\theta_1 + \phi) \sin \theta_3\}$$

$$R = -\frac{1}{2} \{2m_B r_3^2 \sin \theta_2 \cos \theta_2\}$$

$$S = -\frac{1}{2} \{(m_3 + 2m_B)r_1r_3 \cos \theta_1 \sin \theta_2\}$$

$$T = -\frac{1}{2} \{(m_5 + 2m_C)r_4r_5 \sin(\theta_1 + \phi) \cos \theta_3\}$$

$$U = -\frac{1}{2} \{(m_3 + 2m_B)r_1r_3 \sin \theta_1 \cos \theta_2\}$$

$$V = -\frac{1}{2} \{2(m_3 + m_B)r_1^2 \sin \theta_1 \cos \theta_1 + 2(m_5 + m_C)r_4^2 \sin(\theta_1 + \phi) \cos(\theta_1 + \phi)\}.$$

For the four-point type, the entry C becomes

$$C = -\frac{1}{2} \{2m_2r_1^2 + 2(m_3 + m_B)r_1^2 \sin^2 \theta_1 + 2(m_5 + m_C)r_4^2 \sin^2(\theta_1 + \phi)\}.$$

APPENDIX B

We chose $\mathbf{v} = [\theta_1]$, then $\mathbf{u} = [\theta_2, \theta_3]$, and

$$\Phi_{\mathbf{v}} = \begin{bmatrix} r_1 \cos \theta_1 \\ r_4 \cos(\theta_1 + \phi) \end{bmatrix}$$

$$\Phi_{\mathbf{u}} = \begin{bmatrix} 0 & r_3 \cos \theta_2 \\ r_5 \cos \theta_3 & 0 \end{bmatrix}.$$

The determinant of $\Phi_{\mathbf{u}} = -r_3r_5 \cos \theta_2 \cos \theta_3$. In the full stroke, there is no such θ_2 and θ_3 that make the determinant of $\Phi_{\mathbf{u}} = 0$. Therefore this partitioning of $\theta = [\mathbf{u}^T, \mathbf{v}^T]^T$ is suitable.

Therefore

$$\mathbf{M}^{\mathbf{uu}} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

$$\mathbf{M}^{\mathbf{uv}} = \begin{bmatrix} D \\ E \end{bmatrix}$$

$$\mathbf{M}^{\mathbf{vv}} = [D \quad E]$$

$$\mathbf{M}^{\mathbf{v}} = [C]$$

$$\mathbf{N}^{\mathbf{v}} = \begin{bmatrix} P & 0 & Q \\ 0 & R & S \end{bmatrix} \begin{Bmatrix} \theta_2^2 \\ \theta_3^2 \\ \theta_1^2 \end{Bmatrix}$$

$$\mathbf{N}^{\mathbf{u}} = \begin{bmatrix} T & U & V \end{bmatrix} \begin{Bmatrix} \theta_2^2 \\ \theta_3^2 \\ \theta_1^2 \end{Bmatrix}$$

$$\mathbf{Q}^{\mathbf{u}} = \begin{bmatrix} P_C r_5 \sin \theta_3 \\ P_B r_3 \sin \theta_2 \end{bmatrix}$$

$$\mathbf{Q}^{\mathbf{v}} = [P_B r_1 \sin \theta_1 + P_C r_4 \sin(\theta_1 + \phi)]$$

where the elements (e.g. $A, B, C, \dots$) are the same as in Appendix A.