Hopf bifurcation to a short porous journal-bearing system using the Brinkman model: weakly nonlinear stability

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Received 18 April 2001; received in revised form 30 August 2001; accepted 12 October 2001

Abstract

On the basis of the Brinkman model, the weakly nonlinear stability characteristics of short porous journal-bearing systems are presented. By applying the Hopf bifurcation theory, the weakly nonlinear behaviors near the critical stability boundary are predicted. According to results, the onset of oil whirl for porous bearings is a bifurcation phenomenon; it can exhibit supercritical limit cycles or subcritical limit cycles for journal speeds in the vicinity of the bifurcation point. With a fixed permeability parameter, such supercritical limit cycles for journal speeds in excess of the threshold speed are confined to a specific region in the \((\omega, \xi)\) plane; and outside this region subcritical limit cycles exist for journal speeds below the threshold speed. In addition, increasing the value of system parameter, \(S_p\), may change supercritical bifurcation into the more complicated subcritical bifurcation. © 2002 Published by Elsevier Science Ltd.

Keywords: Porous bearings; Journal bearings; Brinkman model; Bleakly nonlinear stability; Hopf bifurcation

1. Introduction

Self-acting porous journal bearings have been investigated for many years because of their low costs and requiring no exterior oil supply. Using the Darcy model (DM), analyses of porous journal bearings were presented by Morgan and Cameron [1], Rouleau [2], Cusano [3], and Murti [4]. Based upon the slip–flow model (SFM) proposed by Beavers and Joseph [5], the effects of velocity slip on the performance of porous journal bearings were considered by Prakash and Vij [6], Rouleau and Steiner [7], Cusano [8] and Bujurke and Naduvinamani [9]. However, Neale and Nader [10] have proposed that a discontinuity of the tangential velocity component across the permeable surface is not permissible. Since the Brinkman model (BM) contains a viscous shear term matching the shearing stress across the fluid/porous matrix interface, the artificial SFM is not needed. On the basis of this BM, lubrication performances of porous journal bearings have been presented by Lin and Hwang [11–13]. It shows that the viscous shear effects increase the load capacity and reduce the coefficient of friction. All of these works concentrate on the static characteristics only.

By applying the DM, stability charts of short porous journal bearings were obtained by Conry and Cusano [14]. Based on the SFM, the dynamic characteristics of porous journal bearings were presented by Chandra et al. [15], Chattopadhyay and Majumdar [16,17] and Kaneko [18]. The effects of velocity slip on the dynamic characteristics were found to be small. On the basis of the BM, the dynamic characteristics of porous journal bearings are predicted by Lin and Hwang [19–21]. According to their results, the viscous shear effects increase the stability threshold speed of the system. However, by linearizing the equations of motion about the equilibrium point, all of the above papers have determined on which combination of parameters the journal becomes unstable and starts whirling. On the other hand, using multiple-scale methods or Hopf bifurcation theory, the weakly nonlinear stability of a non-porous journal bearing has been analyzed by Gardner et al. [22], Meyers [23] and Hollis and Taylor [24]. Since the weakly nonlinear dynamic...
## Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$b$</td>
<td>length of bearing</td>
</tr>
<tr>
<td>$C$</td>
<td>radial clearance, $r_1 - R$</td>
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<tr>
<td>$C_{in}$</td>
<td>damping coefficient, $-(\partial F_\ell/\partial \dot{n})_s$, ($l$, $n=X$, $Y$)</td>
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<tr>
<td>$e$</td>
<td>eccentricity</td>
</tr>
<tr>
<td>$F_X$</td>
<td>fluid-film force in the $X$-direction</td>
</tr>
<tr>
<td>$F_Y$</td>
<td>fluid-film force in the $Y$-direction</td>
</tr>
<tr>
<td>$h$</td>
<td>dimensionless film thickness, $h^*/C=1+\epsilon \cos \theta$</td>
</tr>
<tr>
<td>$H$</td>
<td>thickness of bearing matrix</td>
</tr>
<tr>
<td>$K$</td>
<td>permeability of bearing material</td>
</tr>
<tr>
<td>$K_{in}$</td>
<td>stiffness coefficient, $-(\partial F_\ell/\partial n)_s$, ($l$, $n=X$, $Y$)</td>
</tr>
<tr>
<td>$m$</td>
<td>half of the rotor mass</td>
</tr>
<tr>
<td>$p$</td>
<td>dimensionless film pressure, $p^<em>$ $C^2/\mu \omega^</em> R^2$</td>
</tr>
<tr>
<td>$q$</td>
<td>eigenvalue of the Jacobin matrix</td>
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<tr>
<td>$R$</td>
<td>radius of the journal rotor</td>
</tr>
<tr>
<td>$r_0$</td>
<td>outer radius of porous bearing</td>
</tr>
<tr>
<td>$r_1$</td>
<td>inner radius of porous bearing</td>
</tr>
<tr>
<td>$R$</td>
<td>radius of the journal rotor</td>
</tr>
<tr>
<td>$S_m$</td>
<td>modified Sommerfeld number, $<a href="R/C">\mu \omega^* R/(W/b)</a>^2 \lambda^2$</td>
</tr>
<tr>
<td>$S_p$</td>
<td>system parameter, $S_m/\omega$</td>
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<tr>
<td>$S_{p0}$</td>
<td>system parameter, $S_m/\omega_0$</td>
</tr>
<tr>
<td>$T$</td>
<td>period of the limit cycle</td>
</tr>
<tr>
<td>$W$</td>
<td>steady load on bearing</td>
</tr>
<tr>
<td>$X$</td>
<td>dimensionless coordinate, $\epsilon \cos \psi$</td>
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<tr>
<td>$Y$</td>
<td>dimensionless coordinate, $\epsilon \sin \psi$</td>
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<tr>
<td>$z$</td>
<td>dimensionless coordinate, $z^*/b$</td>
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### Greek symbols

<table>
<thead>
<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td>$\alpha$</td>
<td>parameter appearing in the Brinkman model</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>stability parameter</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$r_0/r_1$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>direction of bifurcation</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>eccentricity ratio, $\epsilon/C$</td>
</tr>
<tr>
<td>$\eta$</td>
<td>real part of eigenvalue $q$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>nondimensional circumferential coordinate</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>length-to-diameter ratio, $b/2R$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>dimensionless time, $\omega^* t$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>permeability parameter, $Kr_1/C^3$</td>
</tr>
<tr>
<td>$\psi$</td>
<td>attitude angle</td>
</tr>
<tr>
<td>$\omega$</td>
<td>dimensionless angular velocity, $(mC/W)^{1/2}\omega^*$</td>
</tr>
<tr>
<td>$\omega_0$</td>
<td>stability threshold speed</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>imaginary part of eigenvalue $q$</td>
</tr>
<tr>
<td>$\Omega_0$</td>
<td>nondimensional whirl frequency, $\Omega(\omega_0)$</td>
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<tr>
<td>$\nabla$</td>
<td>gradient operator</td>
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### Superscript

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<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\cdot$</td>
<td>differentiation with respect to $\tau$</td>
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<tr>
<td>$^*$</td>
<td>the dimensional quantity</td>
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### Subscript

<table>
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<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$s$</td>
<td>the steady-state quantity</td>
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</table>
characteristics of porous journal bearings close to the linear stability boundary are not known, investigation of the bifurcation phenomena of the system using the appropriate model is of interest.

Based upon the Brinkman model, the present study is mainly concerned with the weakly nonlinear dynamics of short porous journal bearings. By applying Hopf bifurcation theory to the system, the weakly nonlinear behaviors in the vicinity of the linear stability boundary are predicted. The onset of oil whirl is shown as a bifurcation phenomenon. The system exhibits a periodic solution at journal speed close to the threshold speed. The regions of supercritical and subcritical stability depend upon the permeability parameter and the steady-state eccentricity ratio. The value of the system parameter can alter the type of bifurcation behavior of the porous journal-bearing system.

2. Summary of Hopf bifurcation theory

The ordinary differential equations of motion of the journal supported in a porous bearing are nonlinear. It is not usually an easy work to evaluate the weakly nonlinear behavior of the porous-bearing system. Hopf bifurcation theory provides a structured formal approach to examine the bifurcation of periodic solutions from the equilibrium position of the system. For details of the theory and application of Hopf bifurcation the reader is referred to Refs. [23–26]. For ready reference, Hopf bifurcation theory is summarized in the following.

Consider a system of differential equations as the following:
\[ \dot{z} = F(z, v), \]  
where \( z \) denotes the state variable and \( v \) is a system parameter (e.g. \( \phi \) in the present problem). Since the eigenvalues of the linearized system are functions of \( v \), it is possible for the eigenvalues to have zero real part for \( v=v_0 \) (critical value). As a result, linear stability theory fails.

To determine stability one needs higher-order terms. On the plot of root locus, the eigenvalues cross the imaginary axis at the critical value. As the system parameter \( v \) is varied through its critical value \( v_0 \), the isolated stationary point \( z=\mathbf{z}_j(v) \) loses linear stability by having a complex conjugate pair of eigenvalues cross into the right half-plane. This case is called a Hopf bifurcation. At bifurcation the solution of the system is usually a periodic behavior (a limit cycle). A Hopf bifurcation may be supercritical or subcritical. For a supercritical bifurcation, the stable limit cycle appears and the stable limit cycle grows as \( v \) is increased past \( v_0 \). When subcritical behavior occurs, the limit cycle exists for \( v<v_0 \) and the limit cycle shrinks on to the equilibrium point as \( v \) is increased to \( v_0 \). In the case of subcritical bifurcation, if the equilibrium point is stable, the limit cycle is necessarily unstable. For the system (1), the conditions that must be met for the Hopf bifurcation theory are:

(a) \( q_1=q_2=\eta(v)+i\Omega(v) \)  
(b) \( \eta(v_0)=0, \Omega(v_0)>0 \)  
(c) \( (\eta'(v_0))_v=\eta'(v_0) \neq 0 \)  
(d) \( \text{Re}(q_j(v_0))<0 \) for \( j=3, 4, \ldots, n \)

where \( q_j(v), j=1, 2, \ldots, n \), are the eigenvalues in order of decreasing real parts. Under these conditions, the solution is usually periodic at bifurcation. Then a parameter, \( \alpha_2 \), indicating the direction of bifurcation exists and is interpreted as follows.

\( \alpha_2 > 0 \): periodic solutions exist for \( v>v_0 \); and  
\( \alpha_2 < 0 \): periodic solutions exist for \( v<v_0 \).

Moreover, stability of the limit cycle can be determined from the stability parameter, \( \gamma_2 \). If

\( \gamma_2 < 0 \): the solutions are orbitally asymptotically stable; and if  
\( \gamma_2 > 0 \): the solutions are orbitally asymptotically unstable.

3. System equations and system parameter

3.1. Equations of motion of the journal

Fig. 1 shows the physical configuration of a porous journal bearing. According to previous works [19–21],
the equations of motion of the journal can be expressed in a nondimensional form. To investigate the nonlinear system behavior, it is necessary to convert the above equations to a first-order system of ordinary differential equations. Using the state variables \( X, Y, \dot{X}, \dot{Y} \), the equations of motion can be written in terms of the state vector, \( x \), as

\[
\frac{dx}{d\tau} = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{F}_x / \omega^2 \\ \dot{F}_y / \omega^2 \end{bmatrix} = f(x, \omega, S_m),
\]

(2)

where \( \dot{\cdot} \) denotes differentiation with respect to \( \tau \) (nondimensional time), \( \omega \) is the nondimensional rotor angular velocity and \( S_m \) is the modified Sommerfeld number. The nondimensional forces \( F_x \) and \( F_y \), acting in the \( X \)- and \( Y \)-directions, respectively, are nonlinear.

3.2. System parameter

The rate at which an eigenvalue crosses the imaginary axis is an important quantity in the application of Hopf bifurcation theory. In calculating this value it is necessary to realize the relationship of the parameters \( \omega, S_m \) and \( \epsilon \), for the porous-bearing system. Altering the nondimensional rotor speed \( \omega \) affects the value of the modified Sommerfeld number \( S_m \); and the steady-state eccentricity ratio \( \epsilon \) is, then, changed. Accordingly a system parameter, \( S_p \), is introduced as:

\[
S_p = \frac{S_m}{\omega}.
\]

(3)

With this definition we shall obtain an equilibrium position for a given rotor speed, \( \omega \), when a particular rotor (i.e. constant \( S_p \)) is considered. Let those eigenvalues that cross the imaginary axis at the threshold speed be denoted by

\[
q(\omega) = \eta(\omega) \pm i\Omega(\omega),
\]

(4)

where \( \eta(\omega_0) = 0 \) and \( \Omega(\omega_0) = \Omega_0 \). And \( \eta(\omega) + i\Omega(\omega) \) denotes the eigenvalue that is continuous extension of \( +i\Omega_0 \). Then the rate at which an eigenvalue crosses the imaginary axis, i.e. the value of

\[
\eta_0 \left( = \frac{d\eta}{d\omega} \right)_{\omega=\omega_0},
\]

can be calculated from the characteristic equation.

4. Application of Hopf bifurcation theory

Although Hopf bifurcation analysis can provide a qualitative prediction for the behavior of the nonlinear system, the application of the formula is very long and tedious due to its involving the second- and third-order derivatives. In order to simplify the problem, we take the short-bearing approximation to predict the bifurcation phenomena close to the critical stability boundary. For a short porous-bearing system using the Brinkman model, the dimensionless pressure distribution is [21]

\[
p = 12\lambda^2 \left( \frac{1}{z^2} - \frac{1}{4} \right) \left( \frac{1 + B - 2\psi(-\epsilon \sin \theta + 2\epsilon \cos \theta)}{A} \right).
\]

(5)

Integrating the pressure yields the film forces on the journal. In order to shorten the contents, the application of Hopf bifurcation theory to the present autonomous, first-order system of ordinary differential equations (2) is briefly stated as follows.

(1) Find the equilibrium positions of the system. The static equilibrium position can be found in terms of \( \epsilon \) and \( \psi \) from \( f=0 \). Denoting the steady-state conditions by a subscript \( s \), the resulting equations for the static equilibrium position are

\[
S_m = \frac{1}{2 \left( \int_{0=0}^{\theta=\pi} p \cos \theta \, d\theta \right)^2 + \left( \int_{0=0}^{\theta=\pi} p \sin \theta \, d\theta \right)^2} \]

(6)

and

\[
\psi_s = \tan^{-1} \left( \frac{\int_{0=0}^{\theta=\pi} p \cos \theta \, d\theta}{\int_{0=0}^{\theta=\pi} p \sin \theta \, d\theta} \right).
\]

(7)

It is noted that Eq. (6) describes the relationship between the modified Sommerfeld and the steady-state eccentricity ratio; and Eq. (7) depicts the locus of the journal center under steady-state conditions.

(2) Find the eigenvalues of the Jacobian matrix \( J(\omega) \). The matrix is given by

\[
J(\omega) = (\nabla_x f)_{x=x_s}.
\]

(8)

The eigenvalues, \( q \), of the Jacobian matrix satisfy the following characteristic equation:

\[
\omega^4 q^4 + \omega^2 (C_{xx} + C_{yy}) q^2 + \left[ \omega^2 (K_{xx} + K_{yy}) + C_{xx} C_{yy} - C_{xy} C_{yy} - (K_{xx} K_{yy} - C_{xy} K_{xx} - C_{xx} K_{yy}) q \right] + (K_{xx} K_{yy} - K_{xy} K_{yx}) = 0.
\]

(9)

(3) Find the stability threshold speed \( \omega_0 \). By applying the Routh–Hurwitz stability criterion to the characteristic equation, the stability threshold speed \( \omega_0 \) can be obtained. If

(a) \( q_1 \) and \( q_2 \) are a conjugate pair for \( \omega \) in an open interval including \( \omega_0 \).
(b) $\Omega(\omega_0)>0$, 
(c) $\eta'(\omega_0) \neq 0$ and 
(d) $\text{Re}[q_i(\omega_0)]<0$ (for $j=3, 4$),

then a Hopf bifurcation occurs.

(4) Find a matrix $P$ to transform $J(\omega)$ into a real canonical form. The matrix $P$ can be obtained as follows:

(a) find the eigenvector $v_i$ of $J(\omega)$ corresponding to the eigenvalue $q_i=\text{II}_0$;
(b) find $r_1$ and $r_4$, which are any set of real four-vectors spanning the union of the (generalized) eigenspaces of $q_3$ and $q_4$ at $\omega=\omega_0$;
(c) normalize $v_i$ so that its first nonvanishing component is 1; and
(d) form the matrix $P$ as

$$P= [\text{Re}(v_i) - \text{Im}(v_i) \quad r_1 \quad r_4].$$  \hspace{1cm} (10)

(5) Perform the change of variables

$$x= x_0 + Py$$  \hspace{1cm} (11)
and find the system $y$

$$\dot{y} = F(y).$$  \hspace{1cm} (12)

The Jacobian matrix $\partial F/\partial y_j(0)$ ($i, j=1, 2, 3, 4$) will have the real canonical form as

$$\begin{bmatrix}
0 & -\Omega_0 & 0 & 0 \\
\Omega_0 & 0 & 0 & 0 \\
0 & 0 & D_{11} & D_{12} \\
0 & 0 & D_{21} & D_{22}
\end{bmatrix},$$  \hspace{1cm} (13)

where $D_{11}, D_{12}, D_{21}$ and $D_{22}$ are the elements of the matrix $D$.

(6) Calculate the following quantities, all to be evaluated at the bifurcation point:

$$a_{11} = \frac{1}{4}\left[ \frac{\partial^2 F_1}{\partial y_1^2} + \frac{\partial^2 F_1}{\partial y_2^2} + \left( \frac{\partial^2 F_2}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2} \right) \right].$$  \hspace{1cm} (14a)

$$a_{02} = \frac{1}{4}\left[ \frac{\partial^2 F_1}{\partial y_1^2} + \frac{\partial^2 F_1}{\partial y_2^2} - 2\frac{\partial^2 F_2}{\partial y_1 \partial y_2} + i\left( \frac{\partial^2 F_2}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2} \right) \right],$$  \hspace{1cm} (14b)

$$-2\frac{\partial^2 F_1}{\partial y_1 \partial y_2},$$  \hspace{1cm} (14c)

$$a_{20} = \frac{1}{4}\left[ \frac{\partial^2 F_1}{\partial y_1^2} + \frac{\partial^2 F_1}{\partial y_2^2} + 2\frac{\partial^2 F_2}{\partial y_1 \partial y_2} + i\left( \frac{\partial^2 F_2}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2} \right) \right]$$  \hspace{1cm} (14d)

and

$$A_{21} = \frac{1}{8}\left[ \frac{\partial^3 F_1}{\partial y_1^3} + \frac{\partial^3 F_1}{\partial y_1 \partial y_2^2} + \frac{\partial^3 F_2}{\partial y_1 \partial y_2^2} + \frac{\partial^3 F_2}{\partial y_2^3} + i\left( \frac{\partial^3 F_2}{\partial y_1^3} + \frac{\partial^3 F_2}{\partial y_1 \partial y_2^2} \right) \right]$$  \hspace{1cm} (14e)

(7) Find $h_{11}^2$ and $h_{20}^2$ for $k=3, 4$:

$$h_{11}^2 = \frac{1}{4}\left[ \frac{\partial^2 F_1}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial y_2^2} \right]$$  \hspace{1cm} (15a)

and

$$h_{20}^2 = \frac{1}{4}\left[ \frac{\partial^2 F_1}{\partial y_1^2} - 2\frac{\partial^2 F_2}{\partial y_2^2} \right],$$  \hspace{1cm} (15b)

where $h_{11}$ and $h_{11}$ are the elements of $h_{11}$, and $h_{20}$ and $h_{20}$ are the elements of $h_{20}$.

Find the two-dimensional vectors $w_{11}$ and $w_{20}$ of the linear system:

$$ Dw_{11} = -h_{11},$$  \hspace{1cm} (16a)

and

$$ (D - 2i\Omega_0 I) w_{20} = -h_{20}. $$  \hspace{1cm} (16b)

Find $A_{1110}$ and $A_{1020}$ for $k=3, 4$:

$$ A_{1110} = \frac{1}{2}\left[ \frac{\partial^2 F_1}{\partial y_1 \partial y_2} + \frac{\partial^2 F_2}{\partial y_2 \partial y_2} + i\left( \frac{\partial^2 F_2}{\partial y_1 \partial y_2} - \frac{\partial^2 F_1}{\partial y_1 \partial y_2} \right) \right] $$  \hspace{1cm} (17a)

and

$$ A_{1020} = \frac{1}{2}\left[ \frac{\partial^2 F_1}{\partial y_1 \partial y_2} - \frac{\partial^2 F_2}{\partial y_2 \partial y_2} + i\left( \frac{\partial^2 F_2}{\partial y_1 \partial y_2} + \frac{\partial^2 F_1}{\partial y_1 \partial y_2} \right) \right]. $$  \hspace{1cm} (17b)

Find $a_{21}$:

$$ a_{21} = A_{21} + \sum_{k=1}^{2}\left( 2A_{1110}w_{11}^k + A_{1020}w_{20}^k \right). $$  \hspace{1cm} (18)

(8) Find the quantity $C_1(0)$:

$$ C_1(0) = \frac{i}{2\Omega_0}\left[ a_{20}a_{11} - 2|a_{11}|^2 - \frac{1}{3}|a_{02}|^2 \right] + \frac{1}{2}a_{21}. $$  \hspace{1cm} (19)

(9) Find the direction of bifurcation $\alpha_2$ and the stability parameter $\gamma_2$:

$$ \alpha_2 = \frac{-\text{Re}[C_1(0)]}{\eta_0} $$  \hspace{1cm} (20)

and

$$ \gamma_2 = 2\text{Re}[C_1(0)]. $$  \hspace{1cm} (21)

(10) Find the period of the limit cycle, $T$, and the periodic solutions (limit cycle), $x(\tau)$. The period can be obtained from

$$ T = \frac{2\pi}{\Omega_0}\left[ 1 + \beta_2 \frac{\omega - \omega_0}{\alpha_2} \right], $$  \hspace{1cm} (22)

where the parameter $\beta_2$ is
\[
\beta_2 = -\frac{\text{Im}[C_1(0)] + \alpha_2 \Omega'(\omega_0)}{\Omega_0}. \tag{23}
\]

The limit cycle is then approximated by
\[
x = x_s + \left(\frac{\omega - \omega_0}{\alpha_2}\right)^{1/2} \text{Re}[e^{2\pi n \sigma T_1}]. \tag{24}
\]

The evaluation of the second- and third-order partial derivatives give stability and frequency information of the limit cycle, just as the first-order partial derivatives give stability and frequency information of the equilibrium point.

5. Results and discussion

Based upon the Brinkman model, the weakly nonlinear dynamic characteristics of short porous journal bearings is considered. By applying Hopf bifurcation theory to the system, the bifurcation behaviors in the vicinity of the linear stability boundary are of interest. In order to show the existence of Hopf bifurcation for short porous bearings, the important quantities, i.e. \(\eta_0\) and \(\Omega_0\), need to be presented. In the following results the chosen values of parameters for the present system are: \(l = 0.5\), \(C/r_1 = 0.001\), \(\beta = 1.2\) and \(\alpha = 1\).

5.1. Existence of Hopf bifurcation for porous bearings

Following the procedure described above, the rate at which an eigenvalue crosses the imaginary axis (\(\eta_0\)), the nondimensional whirl frequency (\(\Omega_0\)), the system parameter (\(S_{p0} = S_p/\omega_0\)) and the stability threshold speed (\(\omega_0\)) can be obtained. Fig. 2 displays the results of \(\eta_0\), \(\Omega_0\), \(S_{p0}\) and \(\omega_0\) versus steady-state eccentricity ratio \(\epsilon_s\) as the value of permeability parameter (\(\Phi\)) approaches zero. Also shown are the results of the short non-porous bearing obtained by Gardner et al. [22]. It is seen that, as \(\Phi\) approaches zero, the values of \(\eta_0\), \(\Omega_0\), \(S_{p0}\) and \(\omega_0\) for the porous-bearing system agree with those of the non-porous bearing. This agreement supports our linear prediction and provides a basis for the use of weakly nonlinear prediction. Moreover, the mainly technical conditions of the Hopf bifurcation theory (\(\Omega_0 > 0\) and \(\eta_0 \neq 0\)) are satisfied. Therefore, the system exists a periodic solution at rotor speed close to the threshold speed. It remains then to determine the direction of bifurcation, the size and the stability of the limit cycle.

5.2. Stability regions

Using Hopf bifurcation analysis, the direction of bifurcation (\(\alpha_2\)) and the stability parameter (\(\gamma_2\)) can be obtained; and then, the regions of subcritical and supercritical stability can be predicted. Fig. 3 shows the stability regions with \(\Phi = 0.001\) for different system parameter \(S_p\) (= \(S_p/\omega_0\)). The curve of \(\omega_0\) represents the neutral stability of the system; and each curve of constant \(S_p\) illustrates the relationship between nondimensional rotor speed \(\omega\) and steady-state eccentricity ratio \(\epsilon_s\). The constant-\(S_p\) lines are useful, since they allow some prediction of system behavior when the nondimensional rotor speed is varied for a particular rotor. In the region of \(\epsilon_s < 0.316\), the direction of bifurcation \(\alpha_2\) and the stability parameter \(\gamma_2\) are found to be: \(\alpha_2 < 0\), \(\gamma_2 > 0\). From the Hopf bifurcation theory subcritical bifurcation occurs at \(\omega = \omega_0\) and, therefore, the bifurcated periodic orbit is unstable. Moreover, no small-amplitude orbits can exist.

Fig. 2. Comparison of \(\eta_0\), \(\Omega_0\), \(S_{p0}\) and \(\omega_0\) using the BM with a non-porous case.

Fig. 3. Stability regions for different \(S_p\) with \(\Phi = 0.001\).
for $\omega > \omega_0$. In the region of $\varepsilon > 0.316$, we have: $\alpha_2 > 0$, $\gamma_2 < 0$. Thus supercritical bifurcation occurs at $\omega > \omega_0$, and the bifurcated periodic orbit is stable. In this figure, the subcritical and supercritical bifurcation regions of a non-porous bearing are also divided by a broken line at $\varepsilon = 0.15$. This result is predicted by Gardner et al. [22] using the multiple-scale method. By comparison with the results of a non-porous bearing, the permeability of a porous bearing yields a larger region of subcritical bifurcation. These results are similar to that of the prediction by Conry and Cusano [14]. Although they did not give any information about bifurcation phenomena, they showed that the stable regions shrunk at increasing values of $\Phi$.

Fig. 4 shows the stability regions with $\Phi = 0.01$ for different system parameter $S_p$. In the region of $\varepsilon < 0.325$, the direction of bifurcation $\alpha_2$ and the stability parameter $\gamma_2$ are found to be: $\alpha_2 < 0$, $\gamma_2 > 0$. Subcritical bifurcation occurs at $\omega < \omega_0$ and, therefore, the bifurcated periodic orbit is unstable from the Hopf bifurcation theory. In addition, no small-amplitude orbits can exist for $\omega > \omega_0$. In the region of $\varepsilon > 0.325$, we have: $\alpha_2 > 0$, $\gamma_2 < 0$. Accordingly, supercritical bifurcation occurs at $\omega > \omega_0$, and the bifurcated periodic orbit is stable. Fig. 5 shows the stability regions with $\Phi = 0.1$ for different values of $S_p$. In the region of $\varepsilon < 0.408$, the direction of bifurcation and the stability parameter are found to be: $\alpha_2 < 0$, $\gamma_2 > 0$. Thus subcritical bifurcation occurs at $\omega < \omega_0$ and the bifurcated periodic orbit is an unstable one. In the region of $\varepsilon > 0.408$, it is found that $\alpha_2 > 0$, $\gamma_2 < 0$; then supercritical bifurcation occurs at $\omega > \omega_0$. And the bifurcated periodic orbit is stable. Fig. 6 indicates the stability regions with $\Phi = 1$ for different $S_p$. In the region of $\varepsilon > 0.82$, the direction of bifurcation and the stability parameter are found to be: $\alpha_2 < 0$, $\gamma_2 > 0$. The system with $S_p = 2$ has a subcritical bifurcation for $\omega < \omega_0$; and the bifurcated periodic orbit is unstable.

For fixed permeability parameter $\Phi$ there is a need to realize how the values of system parameter $S_p$ for different rotors affect the stability threshold speed. Fig. 7 displays the stability threshold speed as a function of $S_p$ for various $\Phi$. It is seen that for low $\Phi$ ($\Phi = 0.001$) the curve of stability threshold speed has a concave point with small system parameter (about $S_p = 0.1$). But for high permeability ($\Phi = 0.01, 0.1$) an increase in the value of $S_p$ results in a higher threshold speed.
From above results, it is shown that the porous-bearing system can exhibit stable limit cycles for journal speeds above the threshold speed. Such supercritical limit cycles for rotor speeds in excess of the threshold speed are confined to a specific region in the \((\omega, \epsilon)\) plane with fixed \(\Phi\); and outside this region subcritical limit cycles exist for rotor speed below the threshold speed. On the whole, the bifurcation may be subcritical or supercritical depending upon the value of steady-state eccentricity ratio with fixed permeability parameter. Moreover, a decrease in the value of \(\Phi\) enlarges the supercritical region, but shrinks the region of subcritical stability.

5.3. Supercritical bifurcation

In order to describe the Hopf bifurcation behavior of the porous-bearing system, Fig. 8 illustrates the supercritical bifurcation diagrams for different \(S_p\) with \(\Phi=0.01\). Stable solutions are shown by solid lines, and unstable solutions by dashed lines. For \(S_p=0.2\) with \(\omega=\omega_0\) (=2.4366) the rotor is stable at the position of steady-state eccentricity ratio. With \(\omega>\omega_0\), the Hopf bifurcation analysis predicts a stable periodic solution of the system; and the supercritical Hopf bifurcation appears close to the bifurcation point. Similar features are predicted for \(S_p=0.1\) and \(S_p=0.06\). It seems that the predicted limit cycles grow as the value of \(S_p\) increases. In other words, with fixed permeability parameter the porous-bearing system possesses a larger limit cycle for larger system parameters, when supercritical Hopf bifurcation occurs.

To examine the qualitative prediction by the bifurcation theory, the predicted limit cycles are compared with those using strongly nonlinear simulation by the Runge–Kutta fourth-order method. According to the prediction by the bifurcation theory (Figs. 4 and 8), above the threshold speed \((\omega>\omega_0)\) a stable limit cycle occurs for the system parameter \(S_p=0.2\). Fig. 9 displays the simulated and predicted limit cycles for \(S_p=0.2\) with \(\Phi=0.01\) and \(\omega=2.4416\). It seems that there is some discrepancy between the simulated and predicted results. The larger limit cycle can be expected from the effect of the higher-order terms of the strong nonlinearities as one moves away from the bifurcation point (as shown in Fig. 8). Although there is some difference in the size of limit cycle, the strongly nonlinear simulation supports the existence of a stable limit cycle for \(S_p=0.2\) with
$\omega > \omega_0$ as established by bifurcation theory. A second initial condition inside the limit cycle is also shown in Fig. 9, for which the journal motion is observed to be stable.

5.4. Subcritical bifurcation

Fig. 10 illustrates the subcritical bifurcation diagrams for different $S_p$ with $\Phi=0.01$. Stable solutions are shown by solid lines, and unstable solutions by dashed lines. For $S_p=0.4$ with $\omega > \omega_0 (=2.5966)$, the equilibrium position of the rotor is unstable. But there is an unstable periodic solution for $\omega < \omega_0$: this is the subcritical behavior close to the bifurcation point predicted by Hopf bifurcation theory. Similar features are observed for $S_p=0.8$ and $S_p=2.0$. It is seen that the predicted limit cycles grow as the value of $S_p$ decreases. Totally, the porous-bearing system possesses a smaller limit cycle for larger system parameter with fixed permeability when subcritical Hopf bifurcation occurs.

To see the subcritical bifurcation behavior predicted by Hopf bifurcation theory, the unstable limit cycles are illustrated for a particular rotor. Fig. 11 shows the unstable limit cycles for $S_p=0.4$ with $\Phi=0.01$. Below the threshold speed ($\omega=2.5896<\omega_0$), the subcritical bifurcation occurs. Other initial conditions are also shown in this figure using the strongly nonlinear simulation. Inside the limit cycle the journal behavior is stable, and outside the limit cycle the motion of the journal is unstable.

5.5. Sensitivity to the system parameter

In order to depict the effect of system parameter $S_p$ on the type of bifurcation behavior (sub- or supercritical), Fig. 12 illustrates the bifurcation diagrams for different $S_p$ with $\Phi=0.01$. It is seen that the value of the system parameter can alter the type of bifurcation behavior of the porous journal-bearing system. For fixed permeability, increasing the value of the system parameter close to the bifurcation point may change the weakly nonlinear behavior from supercritical bifurcation into subcritical bifurcation phenomena.

6. Conclusion

On the basis of the Brinkman model, the study of weakly nonlinear stability characteristics of short porous journal bearings is presented. By applying the Hopf
bifurcation theory, the bifurcation behaviors in the vicinity of the linear stability boundary are predicted. According to the results presented, conclusions on bifurcation phenomena for the short porous-bearing system can be drawn as follows.

The onset of oil whirl for the porous journal-bearing system is a bifurcation phenomenon; it can exhibit supercritical limit cycles or subcritical limit cycles for journal speeds near the bifurcation point. For the occurrence of supercritical bifurcation, a stable whirl orbit appears as soon as the rotor speed exceeds its threshold value. At journal speed below the threshold speed, subcritical limit cycles may predict the stability threshold amplitude of the weakly nonlinear dynamics for the system. Increasing the value of permeability parameter results in a larger region of subcritical bifurcation. For the short porous journal-bearing system is a bifurcation phenomenon; it can exhibit complicated phenomenon of subcritical bifurcation. Since the bifurcation phenomena depicted here are due to the weakly nonlinear effects of equations of motion, the limitations of a purely linear stability analysis are exposed. The weakly nonlinear stability analysis not only gives information to realize the bifurcation phenomena close to the bifurcation point, but also provides information for engineers to use in designing porous journal bearings.

References