Quantum-mechanical algorithms have recently become more and more popular in the field of computation science because they can speed up a computation when compared with classical algorithms. A famous example is the factorizing algorithm discovered by Shor [1]. Another, which is what we intend to deal with in this work, is the quantum search algorithm developed by Grover [2,3]. If there is an unsorted database containing \( N \) items, out of which only one marked item satisfies a given condition, then using Grover’s algorithm one will find the object in \( O(\sqrt{N}) \) quantum-mechanical steps instead of \( O(N) \) classical steps. In much subsequent research it has been pointed out that the Grover algorithm is optimal. Bennett et al. [4] showed that no quantum algorithm can complete a search for an object in better than \( O(\sqrt{N}) \) steps. Boyer et al. [5] gave tight bounds on Grover’s algorithms. Zalka [6] improved the tight bounds to show that Grover’s algorithm is optimal and proposed a further improvement in the algorithm. Basically, Grover’s algorithm consists of three successive unitary operations. They are (1) the Walsh-Hadamard transformation on the qubit \( |0\rangle \) to create an initial state in the superposition of the basis states; (2) the \( \pi \) angle rotation of the marked state; and (3) the inversion of the initial state.

Although Grover [7] also proposed that the Walsh-Hadamard transformation can be replaced by almost any unitary transformation to create the initial state, in this work, we will focus on the phase rotations for the initial state, the marked state, and the states orthogonal to them. Long et al. [8] have shown that, when arbitrary rotations for the marked and the initial states are used instead of the inversions in Grover’s original algorithm, the rotation phases must satisfy certain matching conditions. Galindo and Martin-Delgado [9] recently addressed a more general viewpoint on the phase rotations. They gave four parameters in the Grover kernel since the rotations operate on four states including the initial, the marked, and the two states orthogonal to the former two. As the \( \pi \) rad inversions for the initial and the marked states were considered, Galindo and Martin-Delgado concluded that the other two rotation phases must be equal. In this work, we will also study the four phase rotations in the Grover kernel but derive a general matching condition between the parameters without fixing any of them in advance.

Suppose we have to search for \( M \) objects out of \( N \) unsorted elements, or mathematically,
\( G = \begin{pmatrix} \alpha (\gamma - \delta) \frac{M}{N} & \beta (\gamma - \delta) \frac{\sqrt{M(N-M)}}{N} \\ \alpha (\gamma - \delta) \frac{\sqrt{M(N-M)}}{N} & \beta (\gamma - (\gamma - \delta) \frac{M}{N}) \end{pmatrix} \). 

Note that alternatively the parameters can be written
\[ \alpha = e^{i\theta_1}, \beta = e^{i\phi_2}, \gamma = e^{i\phi_1}, \text{ and } \delta = e^{i\phi_1}, \]
where the phases \( \theta_1, \theta_2, \phi_1, \) and \( \phi_2 \) are for the marked state, the state orthogonal to the marked state, the initial state, and the state orthogonal to the initial state, respectively.

Finally, repeat operation \( m \) times the, and require the probability of finding the marked element \( |w\rangle \) to be greater than some particular value. We then require, say,
\[ p = |\langle w | G^m | s \rangle|^2 > \frac{1}{2}. \]

The amplitude \( |\langle w | G^m | s \rangle|^2 \) is deduced as
\[ |\langle w | G^m | s \rangle|^2 = \frac{M}{N} + (\xi_1 / (\xi_2 - 1) |\langle w | g_1 \rangle | |\langle g_1 | s \rangle|^2, \]
where \( \xi_{1,2} \) and \( |\langle g_1 | s \rangle| \) denote the eigenvalues and the corresponding eigenvectors of \( G \), respectively. The detailed expressions for the eigenvalues and eigenvectors are given by
\[ \xi_{1,2} = e^{i\lambda_{1,2}} = \frac{1}{2} \text{Tr} G + \frac{i}{2} \sqrt{(\text{Tr} G)^2 - 4 \text{Det} G}, \]
where
\[ \text{Tr} G = \frac{M(\alpha - \beta)(\gamma - \delta) + N(\gamma \beta - \alpha \delta)}{N}, \]
\[ \text{Det} G = \alpha \beta \gamma \delta, \]
\[ k = N(\gamma \beta - \alpha \delta) - M(\alpha + \beta)(\gamma - \delta). \]

The eigenvector \( |g_1 \rangle \) should be discussed in detail because, as shown in Eq. (6), it determines whether the requirement (5) can be satisfied. As \( M \ll N \) and \( N \gg 1 \), the eigenvector \( |g_1 \rangle \) can be asymptotically expressed as
\[ |g_1 \rangle \approx \left[ \begin{array}{c} \alpha \delta - \beta \gamma \\ \alpha (\gamma - \delta) \frac{\sqrt{N/M}}{1} \end{array} \right] \sim |w\rangle, \]
and thus \( \langle w | g_1 \rangle \langle g_1 | s \rangle = O(1/\sqrt{N}) \), meaning that the requirement (5) will never be satisfied. Figure 1 shows an example for this case. To avoid Eq. (10), we in turn must have the matching condition
\[ \frac{\alpha}{\beta} = \frac{\gamma}{\delta} \quad \text{or} \quad \theta_1 - \theta_2 = \phi_1 - \phi_2, \]

such that the eigenvector then becomes, in the normalized form
\[ |g_1 \rangle = \left[ \frac{\sqrt{(\alpha - \beta) \delta(\gamma - \delta) \alpha}}{1} \right] \sqrt{2}. \]

We then have, under the matching condition (11),
\[ |\langle w | g_1 \rangle | |\langle g_1 | s \rangle|^2 = \frac{\sqrt{\alpha M + \sqrt{(\gamma - \delta) \delta(\gamma - \delta) \alpha}}}{2} \]
\[ = \sin \frac{m \Delta \lambda}{2}. \]

Now, since
\[ |\xi_1 / (\xi_2 - 1) = e^{i m \Delta \lambda} - 1 = 2 \sin \frac{m \Delta \lambda}{2}, \]
where \( \Delta \lambda = \lambda_1 - \lambda_2 \), the probability is then
\[ p = |\langle w | G^m | s \rangle|^2 = |\sqrt{\alpha M + (e^{i m \Delta \lambda} - 1) \langle w | g_1 \rangle \langle g_1 | s \rangle}|^2 \]
\[ \approx \left| \sin \frac{m \Delta \lambda}{2} \right|^2. \]

Figure 2 shows the \( p \) vs \( m \) diagram under the condition (11).

FIG. 1. Probability \( p \) as a function of the time step \( m \) for \( N = 1000, M = 10, \alpha = e^{i\pi}, \beta = e^{i\pi}, \gamma = e^{i\pi}, \delta = e^{i\pi}, \) and \( \delta = e^{i\pi}. \)
It shows that $p$ can approach unity as $u_m^D l_u^5 (2 j^2 1), j = 1, 2, \ldots$. Taking $\pi$ first, we have

$$m = \left| \frac{\pi}{\Delta \lambda} = \frac{\pi}{2 \sqrt{M/N}} \frac{\alpha \delta}{(\alpha - \beta)(\gamma - \delta)} \right|^{1/2}$$

$$= \frac{\pi}{2 \sqrt{M/N}} \left[ \frac{1}{2 \cos(\phi_1 - \phi_2)} \right]^{1/2}. \quad (13)$$

Eventually, the searching time step will be a minimum when

$$\theta_1 - \theta_2 = \pi,$$

and the corresponding time step is

$$\min(m) = \frac{\pi}{4} \sqrt{M/N}.$$

In Fig. 3, the $p$ vs $m$ diagram is shown under an optimal choice of the parameters.

To summarize, the four parameters $\alpha$, $\beta$, $\gamma$, and $\delta$ are taken into consideration in a generalized Grover’s kernel $G$, and the phase-matching condition for these parameters has been deduced. It is found that the marked elements will never be searched unless $\alpha \delta - \beta \gamma = 0$. That is, the eigenvector $|g_1\rangle$ of the Grover kernel $G$ should not asymptotically coincide with the marked state. The optimal option for the relation between these parameters, however, is

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} = 1 \text{ or } \theta_1 - \theta_2 = \phi_1 - \phi_2 = \pi. \quad (14)$$

The choice of parameters taken in the original Grover operator is only the simplest one, in which $\alpha = \gamma = -1$ and $\beta = \delta = 1$ were used. Long et al. [8] treated the phase rotations for the initial and the marked states alone, and the phase-matching condition is exactly the case of $\alpha = \gamma$ and $\beta = \delta = 1$. Galindo and Martin-Delgado [9], on the contrary, discussed the phase rotations for states orthogonal to the initial and the marked states, and addressed the condition $\alpha = \gamma = -1$ and $\beta = \delta$.


