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Aharonov-Bohm effect in quantum-to-classical correspondence of the Heisenberg principle

De-Hone Lin*
National Center for High-Performance Computing, No. 21, Nan-ke 3rd Road, Hsin-Shi, Tainan County 744, Taiwan

Jee-Gong Chang and Chi-Chuan Hwang
Department of Engineering Science, National Cheng Kung University, Tainan, Taiwan
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The exact energy spectrum and wave function of a charged particle moving in the Coulomb field and Aharonov-Bohm's magnetic flux are solved by the nonintegrable phase factor. The universal formula for the matrix elements of the radial operator $\hat{r}^{\alpha}$ of arbitrary power α is given by an analytical solution. The difference between the classical limit of matrix elements of inverse radius in quantum mechanics and the Fourier components of the corresponding quantity for the pure Coulomb system in classical mechanics is examined in reference to the correspondence principle of Heisenberg. Explicit calculation shows that the influence of nonlocal Aharonov-Bohm effect exists even in the classical limit. The semiclassical quantization rule for systems containing the topological effect is presented in the light of Heisenberg's corresponding principle.

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I. INTRODUCTION

Since the global structure of magnetic flux was discovered about 40 yr ago [1], it had great impact on our comprehension of the foundation of quantum theory [2], and shed light on the understanding of the phenomenon of fractional quantum Hall effect [3,4], superconductivity [3–5], repulsive Bose gases [6], cosmic string [7], $(2+1)$D gravity theories [8], and so forth. The recent development of quantum information and computation has brought great attention to the decoherence process problem. The magnetic flux is a possible candidate to make the coherent state last longer, say, for example, by controlling the magnetic flux and voltage over superconductor loop [9]. Hence, theoretically to understand the mechanism of connection from quantum to classical region of system containing the nonlocal effect of magnetic flux becomes a rather significant topic. Furthermore, because the nonlocal *A-B* effect is purely quantum mechanical, the role of *A-B* effect in the transition from quantum to classical region is an interesting question of fundamental importance. Understanding such an issue, will not only be applicable to practical problems but also help to clarify the fundamental concept.

There are two purposes of this paper. First, we want to evaluate the matrix elements $\langle n_r' q' k' | r^{\alpha} | n_r q k \rangle$ of the radial operator $\hat{r}$ with different power α for a charged particle moving in the Coulomb field and nonlocal *A-B* effect. Such a system is of significance in discussing quantum Hall effect, high-T_c superconductor [4]; and its matrix elements are important in many photoelectronic processes. Second, we want to investigate the quantum-to-classical behavior of the same system's inverse radius in the light of Heisenberg's correspondence principle [10,11], which in recent years, the Heisenberg's correspondence principle has been a study of great interest in the quantum chaos and/or the Rydberg (highly excited hydrogenlike) atom [12–16]. Nevertheless, the discussion of nonlocal effect in the Heisenberg's correspondence principle is still lacking. The paper is organized as follows: In Sec. I, we calculate the energy spectrum and wave functions of a charged particle moving in the Coulomb field and *A-B* magnetic flux, which is centered at the origin and pointing along the z axis (called ABC system), by way of combining the nonintegrable phase factor in Schrödinger equation; and then, apply the wave functions to evaluate the matrix elements $\langle n_r' q' k' | r^{\alpha} | n_r q k \rangle$ of the radial operator $\hat{r}$ with different power α. In Sec. II, for understanding the role and position of nonlocal *A-B* effect in the Heisenberg's principle, the quantum-to-classical correspondence of the inverse radius matrix element is discussed in detail. Explicit calculation shows that the *A-B* effect even in the zeroth order of quantum-mechanical matrix elements of the ABC system still survive, which can be used to obtain the semiclassical quantization rule in the case of the topological effect existence [17]. Furthermore, we show that the first order of matrix element, the quasiclassical case, plays a role when the difference between the radial quantum numbers is large. Our conclusions are summarized in Sec. III.

II. UNIVERSAL FORMULA FOR THE MATRIX ELEMENTS OF THE RADIAL OPERATOR $\hat{r}^{\alpha}$ OF THE ABC SYSTEM

The Schrödinger equation for a charged particle with reduced mass μ moving in a potential of spherical symmetry can be expressed as

$$\left[E - H_0\left(\mathbf{r}, \frac{\hbar}{i}\boldsymbol{\nabla}\right)\right]\Psi^0_{nlk}(\mathbf{r}) = 0, \tag{1}$$

where $H_0 = -\hbar^2\nabla^2/2\mu + V(r)$ is the system Hamiltonian and Ψ^0_{nlk} is the wave function. Due to the spherical symmetry, the angular part of the wave function can be decomposed

*Email address: n00ldh00@nchc.gov.tw

as $\Psi^0_{nlk}(\mathbf{r}) = R_{nl}(r)Y_{lk}(\theta,\varphi)$, where Y_{lk} are the well-known spherical harmonics. Equation (1) in the spherical coordinates can be rewritten as

$$\left\{E-\left[-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{l(l+1)\hbar^2}{2\mu r^2}\right] - V(r)\right\}R_{nl}(r)Y_{lk}(\theta,\varphi)=0. \tag{2}$$

Since any arbitrary number pair (l,k) satisfies the equation, we have

$$\sum_{l=0}^{\infty}\sum_{k=-l}^{l}\left\{E-\left[-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{l(l+1)\hbar^2}{2\mu r^2}\right] - V(r)\right\}R_{nl}(r)Y_{lk}(\theta,\varphi)=0. \tag{3}$$

For a charged particle interacting with the magnetic field, the wave function is different from the original one by a global nonintegrable phase factor [19,20]

$$\Psi_{nlk}(\mathbf{r})=\Psi^0_{nlk}(\mathbf{r})\exp\left\{\frac{ie}{\hbar c}\int_P^{\mathbf{r}}\mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}'\right\}, \tag{4}$$

where we have used the vector potential $\mathbf{A}(\mathbf{r}')$ to describe the magnetic field and P to represent the nonintegrable phase of the wave function which depends on the path [20]. For the A-B effect under consideration, the vector potential can be described by

$$\mathbf{A}(\mathbf{r})=\frac{1}{2}B\rho\hat{e}_\varphi \quad (\rho<\epsilon), \tag{5}$$

$$\mathbf{A}(\mathbf{r})=\frac{1}{2}B\frac{\epsilon^2}{\rho}\hat{e}_\varphi=\frac{\Phi}{2\pi\rho}\hat{e}_\varphi \quad (\rho>\epsilon), \tag{6}$$

where $\rho=\sqrt{x^2+y^2}$ is the two-dimensional radial length, $\hat{e}_\varphi$ is the unit vector of coordinate φ, ϵ is the radius of the region where magnetic field exists, and Φ is the magnetic flux given by $\Phi=\pi\epsilon^2 B$. The associated magnetic field lines are confined within a tube along the z axis. Along the free region of magnetic field, the path-dependent nonintegrable phase factor is given by $\exp[-i\mu_0\int_P^\tau d\tau'\dot{\varphi}(\tau')]$, where $\dot{\varphi}=d\varphi/d\tau$ and $\mu_0=-2eg/\hbar c$ is a dimensionless number defined by $\Phi=4\pi g$. The minus sign is a matter of convention. According to the discussion of Ref. [20], only the loops of the phase factor need to be considered, the description of the electromagnetic phenomenon is then complete. Therefore, the integral $\int_P^\tau d\tau'\dot{\varphi}(\tau')$ can be written as $(2\pi\tilde{n}+\varphi)$, which satisfies the topological homotopy class. The magnetic interaction is therefore purely topological. The nonintegrable phase factor becomes $\exp\{-i\mu_0(2\pi\tilde{n}+\varphi)\}$. The influence of the magnetic flux in the wave function Ψ^0 can be considered by noting the relation between the associated Legendre polynomial $P^\mu_\nu(z)$ [21]:

$$P^k_l(\cos\theta)=(-1)^k\frac{\Gamma(l+k+1)}{\Gamma(l+1)}\left(\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)^k P^{(k,k)}_{l-k}(\cos\theta), \tag{7}$$

as well as Poisson's summation formula

$$\sum_{k=-\infty}^{\infty}f(k)=\int_{-\infty}^{\infty}dx\sum_{n=-\infty}^{\infty}e^{2\pi nxi}f(x), \tag{8}$$

(e.g., p. 469 in Ref. [22]). We find that Eq. (3) turns into [21,23,18]

$$\sum_{q=0}^{\infty}\sum_{k=-\infty}^{\infty}\left\{E-\left[-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{\tilde{l}(\tilde{l}+1)\hbar^2}{2\mu r^2}\right] - V(r)\right\}R_{n_r\tilde{l}}(r)\mathcal{Y}_{qk}(\theta,\varphi)=0, \tag{9}$$

where we have defined $\tilde{l}=q+|k+\mu_0|$; and the angular part is given by

$$\mathcal{Y}_{qk}(\theta,\varphi)=\sqrt{\frac{(2\tilde{l}+1)}{4\pi}\frac{\Gamma(q+1)\Gamma(\tilde{l}+|k+\mu_0|+1)}{\Gamma^2(\tilde{l}+1)}}\times\left(\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right)^{|k+\mu_0|}P_q^{(|k+\mu_0|,|k+\mu_0|)}(\cos\theta)e^{ik\varphi}. \tag{10}$$

We see that the influence of the A-B effect in the radial wave function is completely determined by replacing the integer quantum number l by the fractional one $\tilde{l}$. With the help of the orthogonal properties of the angular part [21],

$$\int_{-1}^{1}dz(1-z)^\mu(1+z)^\nu P_m^{(\mu,\nu)}(z)P_n^{(\mu,\nu)}(z) = \frac{2^{\mu+\nu+1}}{\mu+\nu+2n+1}\frac{\Gamma(\mu+n+1)\Gamma(\nu+n+1)}{n!\Gamma(\mu+\nu+n+1)}\delta_{m,n} \tag{11}$$

and

$$\int_0^{2\pi}d\varphi e^{i(k-k')\varphi}=2\pi\delta_{k,k'}, \tag{12}$$

it is easy to find that the radial wave function with the fixed quantum number (q,k) satisfies the wave equation

$$\left\{E-\left[-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{\tilde{l}(\tilde{l}+1)\hbar^2}{2\mu r^2}\right]-V(r)\right\}R_{n_r\tilde{l}}(r) =0. \tag{13}$$

In the following contents, due to the importance of hydrogen atom, we consider a charged particle moving in Coulomb field $V(r)=-e^2/r$ and A-B effect with e the charge of particle. The equation is simplified by letting $u_{\tilde{l}}=r\ R_{n_r\tilde{l}}$; and then yields

$$\frac{d^2u_{\tilde{l}}}{dr^2}+\left[2E+\frac{2}{r}-\frac{\tilde{l}(\tilde{l}+1)}{r^2}\right]u_{\tilde{l}}=0, \tag{14}$$

where we let $\hbar=\mu=e=1$ for simplicity. The equation has two singularities located at $r=0,\infty$. In the limit $r\rightarrow 0$, there is the asymptotic form

$$\frac{d^2u_{\tilde{l}}}{dr^2}-\frac{\tilde{l}(\tilde{l}+1)}{r^2}u_{\tilde{l}}=0. \tag{15}$$

It is easy to find $u_{\tilde{l}}(r)\propto r^{\tilde{l}+1},r^{-\tilde{l}}$. The acceptable solution in physics is $u_{\tilde{l}}=r\ R_{n,\tilde{l}}\propto r^{\tilde{l}+1}$. On the other hand, in the limit $r\rightarrow\infty$, Eq. (14) reduces to

$$\frac{d^2u_{\tilde{l}}}{dr^2}+2Eu_{\tilde{l}}=0. \tag{16}$$

We have $u_{\tilde{l}}(r)\propto\exp\{-\sqrt{-2E}r\}$, while the problem is in the case of bound states. Summarizing all this yields a suitable solution for $u_{\tilde{l}}(r)=r^{\tilde{l}+1}e^{-\beta r}v_{\tilde{l}}(r)$, where $\beta=\sqrt{-2E}$ is defined. Inserting this into Eq. (14) yields

$$r\frac{d^2v_{\tilde{l}}}{dr^2}+[2(\tilde{l}+1)-2\beta r]\frac{dv_{\tilde{l}}}{dr}-2[(\tilde{l}+1)\beta-1]v_{\tilde{l}}=0. \tag{17}$$

A further reduction can be obtained by letting $\xi=2\beta r$, which yields

$$\xi\frac{d^2v_{\tilde{l}}}{d\xi^2}+[2(\tilde{l}+1)-\xi]\frac{dv_{\tilde{l}}}{d\xi}-\left[(\tilde{l}+1)-\frac{1}{\beta}\right]v_{\tilde{l}}=0. \tag{18}$$

The equation is the confluent hypergeometric equation (p. 268 of Ref. [22]). An acceptable solution is given by confluent hypergeometric function $v=F((\tilde{l}+1)-1/\beta,2(\tilde{l}+1);\xi)$ with an additional important demand $(\tilde{l}+1)-1/\beta=-n_r$ ($n_r=0,1,2,\ldots$) for obtaining a physical reasonable polynomial solution. Accordingly, we find the energy spectra of the ABC system,

$$E_{n_r,q,k}=-\frac{\mu e^4/\hbar^2}{2(n_r+q+|k+\mu_0|+1)^2}, \tag{19}$$

for $0\leqslant n_r\leqslant\infty$, $0\leqslant q\leqslant\infty$, and $-\infty\leqslant k\leqslant\infty$ which are the correct spectra for the ABC system (e.g., Refs. [18,21,23]). The normalized wave functions are given by

$$R_{n_r,q,k}(r)=\left(\frac{2}{\tilde{n}^2a^{3/2}}\right)\frac{1}{\Gamma(2\tilde{l}+2)}\sqrt{\frac{\Gamma(\tilde{n}+\tilde{l}+1)}{n_r!}}\left(\frac{2r}{\tilde{n}a}\right)^{\tilde{l}}\times e^{-r/\tilde{n}a}F\left(-n_r,2\tilde{l}+2;\frac{2r}{\tilde{n}a}\right), \tag{20}$$

where we have defined the notation $\tilde{n}=(n_r+q+|k+\mu_0|+1)$ and the Bohr radius $a=\hbar^2/\mu e^2$. The wave functions can be applied to evaluate the matrix elements $\langle n_r'q'k'|r^\alpha|n_rqk\rangle$ of the radial operator $\hat{r}$ with different power α. Using Eqs. (11) and (12), we have the orthonormal property

$$\int_0^{2\pi}\int_0^{\pi}d\Omega\mathcal{Y}^*_{q'k'}(\theta,\varphi)\mathcal{Y}_{qk}(\theta,\varphi)=\delta_{q,q'}\delta_{k,k'}, \tag{21}$$

with the solid angle $d\Omega=\sin\theta d\theta d\varphi$. By using this relation, the matrix element can be transformed into a form such as

$$\begin{aligned}\langle n_r'qk|r^\alpha|n_rqk\rangle &=\left(\frac{4}{\tilde{n}'^2\tilde{n}^2a^3}\right)\frac{1}{\Gamma^2(2\tilde{l}+2)}\sqrt{\frac{\Gamma(\tilde{n}'+\tilde{l}+1)\Gamma(\tilde{n}+\tilde{l}+1)}{n_r'!n_r!}}\\&\times\left(\frac{2}{\tilde{n}'a}\right)^{\tilde{l}}\left(\frac{2}{\tilde{n}a}\right)^{\tilde{l}}\int_0^\infty drr^{2\tilde{l}+\alpha+2}\exp\left\{-\frac{r}{a}\left[\frac{\tilde{n}'+\tilde{n}}{\tilde{n}'\tilde{n}}\right]\right\}\\&\times F\left(-n_r',2\tilde{l}+2;\frac{2r}{\tilde{n}'a}\right)F\left(-n_r,2\tilde{l}+2;\frac{2r}{\tilde{n}a}\right),\end{aligned} \tag{22}$$

where the abbreviation $\tilde{n}'=(n_r'+q+|k+\mu_0|+1)$ is used. To obtain the matrix elements we have to perform the integration in Eq. (22). It can be simplified by the relation between the confluent hypergeometric function and the generalized Laguerre polynomials (p. 287 of Ref. [22])

$$F(-n_1,\mu_1+1;z)=\frac{n_1!\Gamma(\mu_1+1)}{\Gamma(n_1+\mu_1+1)}L^{\mu_1}_{n_1}(z). \tag{23}$$

The integral is therefore transformed into a new form

$$\begin{aligned}\int_0^\infty dzz^we^{-\lambda z}&F(-n_1,\mu_1+1;k_1z)F(-n_2,\mu_2+1;k_2z)\\&=\frac{n_1!\Gamma(\mu_1+1)n_2!\Gamma(\mu_2+1)}{\Gamma(n_1+\mu_1+1)\Gamma(n_2+\mu_2+1)}\\&\times\int_0^\infty dzz^we^{-\lambda z}L^{\mu_1}_{n_1}(k_1z)L^{\mu_2}_{n_2}(k_2z).\end{aligned} \tag{24}$$

A further reduction can be obtained by using the polynomial representation of the generalized Laguerее function (p. 240 of Ref. [22])

$$L^{\mu_1}_{n_1}(z)=\sum_{q=0}^{n_1}(-1)^q\binom{n_1+\mu_1}{n_1-q}\frac{z^q}{q!}, \tag{25}$$

which turns Eq. (24) into

$$\int_0^\infty dz z^w e^{-\lambda z} F(-n_1,\mu_1+1;k_1 z)F(-n_2,\mu_2+1;k_2 z)$$
$$=\frac{n_1!\Gamma(\mu_1+1)n_2!\Gamma(\mu_2+1)}{\Gamma(n_1+\mu_1+1)\Gamma(n_2+\mu_2+1)}$$
$$\times\sum_{q=0}^{n_1}\left\{(-1)^q\binom{n_1+\mu_1}{n_1-q}\frac{k_1^q}{q!}\right.$$
$$\left.\times\int_0^\infty z^{w+q}e^{-\lambda z}L_{n_2}^{\mu_2}(k_2 z)dz\right\}. \tag{26}$$

The integral can be performed by the formula (p. 850 of Ref. [24])

$$\int_0^\infty t^\beta e^{-vt}L_n^c(t)dt$$
$$=\frac{\Gamma(\beta+1)\Gamma(c+n+1)}{n!\Gamma(c+1)}v^{-\beta-1}F(-n,\beta+1;c+1;1/v), \tag{27}$$

where $F(a,b;c;z)$ is the hypergeometric function. We complete the integral and obtain the matrix for the radial operator $\hat{r}$ with a general power α of ABC system

$$\langle n_r' qk|r^\alpha|n_r qk\rangle=\frac{1}{4}\left(\frac{4}{\tilde{n}'\tilde{n}}\right)^{\tilde{l}+2}\frac{a^\alpha}{\Gamma(2\tilde{l}+2)}\sqrt{\frac{n_r'!\Gamma(\tilde{n}+\tilde{l}+1)}{n_r!\Gamma(\tilde{n}'+\tilde{l}+1)}}\sum_{i=0}^{n_r'}\left\{\frac{(-1)^i}{i!}\left(\frac{2}{\tilde{n}'}\right)^i\left[\frac{\Gamma(\tilde{n}'+\tilde{l}+1)}{\Gamma(n_r'-i+1)\Gamma(2\tilde{l}+i+2)}\right]\right.$$
$$\left.\times\Gamma(2\tilde{l}+\alpha+i+3)\left(\frac{\tilde{n}'+\tilde{n}}{\tilde{n}'\tilde{n}}\right)^{-2\tilde{l}-\alpha-i-3}F\left(-n_r,2\tilde{l}+\alpha+i+3;2\tilde{l}+2;\frac{2\tilde{n}'}{\tilde{n}'+\tilde{n}}\right)\right\}. \tag{28}$$

For the special case of mean value, with $\alpha=-1$ and $\tilde{n}'=\tilde{n}=(n_r+q+|k+\mu_0|+1)$, it is found to be

$$\left\langle n_r qk\left|\frac{1}{r}\right|n_r qk\right\rangle=\frac{1}{\tilde{n}^2 a}, \tag{29}$$

which is a generalization of expectation value of Coulomb system [25,26]. Similarly, we can find the other terms

$$\left\langle n_r qk\left|\frac{1}{r^2}\right|n_r qk\right\rangle=\frac{1}{\tilde{n}^3 a^2(\tilde{l}+1/2)} \tag{30}$$

and

$$\left\langle n_r qk\left|\frac{1}{r^3}\right|n_r qk\right\rangle=\frac{1}{(\tilde{n}^3 a^3)\tilde{l}(\tilde{l}+1/2)(\tilde{l}+1)}. \tag{31}$$

We see that the mean value is greatly influenced by the nonlocal effect of the magnetic flux although such effect has no significance in classical mechanics.

III. QUANTUM-TO-CLASSICAL CORRESPONDENCE OF MATRIX ELEMENTS FOR THE INVERSE RADIUS OF THE *ABC* SYSTEM IN THE HEISENBERG PRINCIPLE

The Heisenberg's correspondence principle described a relation between the matrix elements of quantum mechanics and the Fourier components of the corresponding classical mechanical quantity. This was concluded by Heisenberg himself from the observation that when quantum numbers are large the matrix elements $\langle\Psi_{n+m}(t)|f|\Psi_n(t)\rangle$ of quantum mechanics are approximated by the mth Fourier component f_m of a classical mechanical quantity $f(t)$ [10]. Explicitly, it can be expressed as

$$\langle\Psi_{n+m}(t)|f|\Psi_n(t)\rangle$$
$$=\langle n+m|f|n\rangle\exp\{i(E_{n+m}-E_n)t/\hbar\}$$
$$\approx f_m(n)\exp\{im\omega(n)t\}, \tag{32}$$

where $\omega(n)$ is the classical frequency, the classical actions J in the Fourier component $f_m(n)$, and $\omega(n)$ are all quantized as $J=n\hbar$. In the following content, owing to the exclusive importance of hydrogen atom in quantum mechanics, we discuss the influence of nonlocal A-B effect on matrix elements of inverse radius of Coulomb system in the classical limit, and then compares it with the Fourier component of the corresponding quantity for the pure Coulomb system. It is shown that the difference is manifestly caused by the nonlocal A-B effect. Moreover, the classical limit of the former gives the semiclassical quantization rule for ABC system in the light of the Heisenberg's principle.

In the polar coordinates, a classical charged particle moving in the pure Coulomb field is described by

$$r=a_0(1-e\cos\psi), \tag{33}$$

where a_0 is the semimajor axis of elliptic orbit, e is the eccentricity, and ψ is the eccentric anomaly which satisfies the Kepler equation [27]:

$$\omega t=\psi-e\sin\psi. \tag{34}$$

By using the Bessel-Kapteyn formula, the Fourier expansion of the inverse radius of Kepler orbit is given by [28,29]

$$\frac{1}{r}=\frac{1}{a_0(1-e\cos\psi)}=\frac{1}{a_0}\sum_{m=-\infty}^{\infty}J_m(me)\exp\{im\omega t\}, \tag{35}$$

with the Bessel function $J_0(0)=1$. One gets the mth Fourier component $J_m(me)/a_0$ which has semiclassical representation in the spirit of Heisenberg's principle,

$$\frac{J_m(me)}{a_0}\rightarrow\frac{J_m(m\sqrt{1-l^2/n^2})}{n^2 a}, \tag{36}$$

where n is the principle quantum number of hydrogen atom; l is the quantum number of angular momentum; a is the Bohr radius as in Sec. II, and the eccentricity e is replaced with $\sqrt{1-l^2/n^2}$ semiclassically. In order to understand the influence of the A-B effect in the classical limit, and then compare it with Eq. (36), we recall the matrix elements $\langle n_r'qk|1/r|n_rqk\rangle$ from Eq. (28),

$$\left\langle n_r'qk\left|\frac{1}{r}\right|n_rqk\right\rangle=\frac{1}{4}\left(\frac{4}{\tilde{n}'\tilde{n}}\right)^{\tilde{l}+2}\frac{1}{a}\sqrt{\frac{n_r'!\,\Gamma(\tilde{n}+\tilde{l}+1)}{n_r!\,\Gamma(\tilde{n}'+\tilde{l}+1)}}\sum_{i=0}^{n_r'}\frac{(-1)^i}{i!}\left(\frac{2}{\tilde{n}'}\right)^i\frac{\Gamma(\tilde{n}'+\tilde{l}+1)}{\Gamma(n_r'-i+1)}\left(\frac{\tilde{n}'\tilde{n}}{\tilde{n}'+\tilde{n}}\right)^{2\tilde{l}+i+2}$$
$$\times\sum_{j=0}^{n_r}\frac{\Gamma(2\tilde{l}+i+j+2)\Gamma(-n_r+j)}{\Gamma(2\tilde{l}+j+2)\Gamma(-n_r)\Gamma(2\tilde{l}+i+2)\Gamma(j+1)}\left(\frac{2\tilde{n}'}{\tilde{n}'+\tilde{n}}\right)^j, \tag{37}$$

where we have used the property of the hypergeometric function

$$F(-m,b;c;z)=\sum_{n=0}^{m}\frac{(-m)_n(b)_n}{(c)_n}\frac{z^n}{n!}. \tag{38}$$

Here $-m$ is a negative integer and the notation $(c)_n=\Gamma(n+c)/\Gamma(c)$ (p. 3, p. 39 of Ref. [22]). For our purpose, the formula is transformed into a new form

$$\left\langle n_r'qk\left|\frac{1}{r}\right|n_rqk\right\rangle=\left[\frac{\Gamma(\tilde{n}'-\tilde{l})}{2\tilde{n}'\Gamma(\tilde{n}'+\tilde{l}+1)}\left(\frac{2}{\tilde{n}'a}\right)^3\right]^{1/2}\left[\frac{\Gamma(\tilde{n}-\tilde{l})}{2\tilde{n}\Gamma(\tilde{n}+\tilde{l}+1)}\left(\frac{2}{\tilde{n}a}\right)^3\right]^{1/2}$$
$$\times a^2 4^{\tilde{l}}\frac{(\tilde{n}'\tilde{n})^{\tilde{l}+2}}{(\tilde{n}'+\tilde{n})^{2\tilde{l}+2}}\sum_{s=0}^{n_r}\frac{(-1)^s\Gamma(\tilde{n}'+\tilde{l}+s+1)}{\Gamma(\tilde{n}-\tilde{l}-s)\Gamma(\tilde{n}'-\tilde{n}+s+1)\Gamma(s+1)}\left(\frac{\tilde{n}'-\tilde{n}}{\tilde{n}'+\tilde{n}}\right)^{\tilde{n}'-\tilde{n}+2s}, \tag{39}$$

for $n_r'>n_r$ which can be easily treated when $n_r', n_r, \tilde{l}$ are all large. The equivalence between Eqs. (37) and (39) can be proven by a rather long but straightforward calculation. Generally speaking, comparing this equation with Eq. (36), we see that the magnitude between matrix element and the Fourier component is quite different. To find their correspondence, let us define the difference $m=\tilde{n}'-\tilde{n}$ which turns Eq. (39) into

$$\left\langle n_r'qk\left|\frac{1}{r}\right|n_rqk\right\rangle=\left[\frac{\Gamma(\tilde{n}+m-\tilde{l})}{2\tilde{n}'\Gamma(\tilde{n}+m+\tilde{l}+1)}\left(\frac{2}{(\tilde{n}+m)a}\right)^3\right]^{1/2}\left[\frac{\Gamma(\tilde{n}-\tilde{l})}{2\tilde{n}\Gamma(\tilde{n}+\tilde{l}+1)}\left(\frac{2}{\tilde{n}a}\right)^3\right]^{1/2}$$
$$\times a^2 4^{\tilde{l}}\frac{[(\tilde{n}+m)\tilde{n}]^{\tilde{l}+2}}{(2\tilde{n}+m)^{2\tilde{l}+2}}\sum_{s=0}^{n_r}\frac{(-1)^s\Gamma(\tilde{n}+m+\tilde{l}+s+1)}{\Gamma(\tilde{n}-\tilde{l}-s)\Gamma(m+s+1)\Gamma(s+1)}\left(\frac{m}{2\tilde{n}+m}\right)^{m+2s}. \tag{40}$$

As $\tilde{n},\tilde{l},\tilde{n}-\tilde{l}$ all have large values, the expression can be simplified. With the help of Γ function $\Gamma(n+\alpha)/\Gamma(\alpha)=\alpha(\alpha+1)\cdots(\alpha+n-1)$, the ratio of two terms in the summation is reduced as follows:

$$\frac{\Gamma(\tilde{n}+m+\tilde{l}+s+1)}{\Gamma(\tilde{n}-\tilde{l}-s)}$$
$$=\frac{\Gamma(\tilde{n}+m+\tilde{l}+1)}{\Gamma(\tilde{n}-\tilde{l})}\prod_{\alpha=1}^{s}(\tilde{n}+m+\tilde{l}+\alpha)(\tilde{n}-\tilde{l}-\alpha)$$
$$=\frac{\Gamma(\tilde{n}+m+\tilde{l}+1)}{\Gamma(\tilde{n}-\tilde{l})}(\tilde{n}^2-\tilde{l}^2)^s\prod_{\alpha=1}^{s}\left(1+\frac{m+\alpha}{\tilde{n}+\tilde{l}}\right)\times\left(1-\frac{\alpha}{\tilde{n}-\tilde{l}}\right)$$
$$\approx\frac{\Gamma(\tilde{n}+m+\tilde{l}+1)}{\Gamma(\tilde{n}-\tilde{l})}(\tilde{n}\tilde{e})^{2s}\left[1+\frac{s(2m+s+1)}{2(\tilde{n}+\tilde{l})}-\frac{s(s+1)}{2(\tilde{n}-\tilde{l})}\right], \quad (41)$$

where we have defined the modified eccentricity $\tilde{e}=\sqrt{1-\tilde{l}^2/\tilde{n}^2}$. On the other hand, keeping to the first order, another factor in summation has the approximation

$$\left(\frac{m}{2\tilde{n}+m}\right)^{m+2s}\approx\left(\frac{m}{2\tilde{n}}\right)^{m+2s}\left[1-\frac{m}{2\tilde{n}}(2s+m)\right]. \quad (42)$$

Thus, the summation in Eq. (40) has the approximation

$$\sum_{s=0}^{n_r}\frac{(-1)^s\Gamma(\tilde{n}+m+\tilde{l}+s+1)}{\Gamma(\tilde{n}-\tilde{l}-s)\Gamma(m+s+1)\Gamma(s+1)}\left(\frac{m}{2\tilde{n}+m}\right)^{m+2s}$$
$$\approx\frac{\Gamma(\tilde{n}+m+\tilde{l}+1)}{\Gamma(\tilde{n}-\tilde{l})}\frac{1}{(\tilde{n}\tilde{e})^m}\sum_{s=0}^{n_r}\frac{(-1)^s}{\Gamma(m+s+1)\Gamma(s+1)}\times\left(\frac{m\tilde{e}}{2}\right)^{2s+m}[1+\varepsilon(m,s)], \quad (43)$$

with the first-order correction

$$\varepsilon(m,s)=-\frac{m}{2\tilde{n}}(2s+m)+\frac{s(2m+s+1)}{2(\tilde{n}+\tilde{l})}-\frac{s(s+1)}{2(\tilde{n}-\tilde{l})}. \quad (44)$$

Inserting Eq. (43) into Eq. (40), we find

$$\left\langle n_r'qk\left|\frac{1}{r}\right|n_rqk\right\rangle=\frac{1}{\tilde{n}^2a}\frac{2^{2\tilde{l}+2}}{(\tilde{n}\tilde{e})^m}\frac{(\tilde{n}+m)^{\tilde{l}}\tilde{n}^{\tilde{l}+2}}{(2\tilde{n}+m)^{2\tilde{l}+2}}\times\sqrt{\frac{\Gamma(\tilde{n}-\tilde{l}+m)\Gamma(\tilde{n}+\tilde{l}+m+1)}{\Gamma(\tilde{n}-\tilde{l})\Gamma(\tilde{n}+\tilde{l}+1)}}\times\sum_{s=0}^{n_r}\frac{(-1)^s}{\Gamma(m+s+1)\Gamma(s+1)}\left(\frac{m\tilde{e}}{2}\right)^{2s+m}\times[1+\varepsilon(m,s)]. \quad (45)$$

The square root is simplified with the following procedure:

$$\sqrt{\frac{\Gamma(\tilde{n}-\tilde{l}+m)\Gamma(\tilde{n}+\tilde{l}+m+1)}{\Gamma(\tilde{n}-\tilde{l})\Gamma(\tilde{n}+\tilde{l}+1)}}=\prod_{\alpha=1}^{m}(\tilde{n}+\tilde{l}+\alpha)(\tilde{n}-\tilde{l}+\alpha-1)\approx(\tilde{n}\tilde{e})^m\left[1+\frac{m(m+1)}{4(\tilde{n}+\tilde{l})}+\frac{m(m-1)}{4(\tilde{n}-\tilde{l})}\right]. \quad (46)$$

Finally, when the quantum number $\tilde{n},\tilde{l},\tilde{n}-\tilde{l}$ are much greater than 1, we obtain the matrix element $\langle n_r'qk|1/r|n_rqk\rangle$ for *ABC* system

$$\left\langle n_r'qk\left|\frac{1}{r}\right|n_rqk\right\rangle=\frac{1}{\tilde{n}^2a}[1+\tilde{\varepsilon}(m)]\sum_{s=0}^{n_r}\frac{(-1)^s}{\Gamma(m+s+1)\Gamma(s+1)}\left(\frac{m\tilde{e}}{2}\right)^{2s+m}\times[1+\varepsilon(m,s)], \quad (47)$$

where the small quantity $\tilde{\varepsilon}$ is given by

$$\tilde{\varepsilon}(m)=-\frac{m}{\tilde{n}}+\frac{m(m+1)}{4(\tilde{n}+\tilde{l})}+\frac{m(m-1)}{4(\tilde{n}-\tilde{l})}. \quad (48)$$

Since the quantum number $n_r=\tilde{n}-\tilde{l}-1$, the series in Eq. (47) is exactly the former $(\tilde{n}-\tilde{l}-1)$ terms of Bessel function, that is,

$$\sum_{s=0}^{\tilde{n}-\tilde{l}-1}\frac{(-1)^s}{\Gamma(m+s+1)\Gamma(s+1)}\left(\frac{m\tilde{e}}{2}\right)^{2s+m}\approx J_m(m\tilde{e}). \quad (49)$$

It is easy to check the absolute error of using the truncated Bessel function involving only the first $(\tilde{n}-\tilde{l}-1)$ terms to represent the function itself is less than the absolute value of the $(\tilde{n}-\tilde{l})$th term. Therefore, up to the zeroth order, we have the matrix element

$$\left\langle n_r'qk\left|\frac{1}{r}\right|n_rqk\right\rangle\approx\frac{1}{\tilde{n}^2a}J_m(m\tilde{e}). \quad (50)$$

We note that the influence of the nonlocal *A*-*B* effect even in the classical limit of matrix element still exists. Comparing this with Eq. (36), the principal quantum number n is changed by the *A*-*B* effect and turns into $\tilde{n}$ which characterizes the modified circular Bohr orbit [30]. According to the correspondence principle of Heisenberg, the semiclassical quantization rule in the case of system containing the *A*-*B* effect is given by $J_k=\tilde{n}_k\hbar$ in which $\tilde{n}_k$ is a fractional number [18,31]. Another interesting outcome can be found in Eq. (48) is that the zero-order approximation is effective only while m is much less than $\tilde{n},\tilde{l}$,and $\tilde{n}-\tilde{l}$. Hence, the first-order correction cannot be neglected, while $m \approx \tilde{n} - \tilde{l}$. In this case, the correspondence principle of Heisenberg gradually deviates the result of quantum mechanics.

IV. CONCLUSION

The connection between quantum mechanics and classical mechanics has been of great interest since quantum mechanics was conceived in the early stages of last century. Various forms of correspondence was proposed by the predecessors who mastered or worked in the field earlier. Understanding of correspondence is not just on the theoretical significance but also has the practical interest in applications of questions such as the quantization of classical chaotic system, the process of decoherence, and the research of highly excited atom. In this paper, we have given the matrix elements of the radial operator $\hat{r}$ with different power α for a charged particle moving in the Coulomb field and nonlocal *A-B* effect. The quantum-to-classical correspondence of Heisenberg for the inverse radius $1/r$ is discussed in accordance with the classical limit of matrix elements. The dominant main purpose of this discussion is to comprehend a possible behavior and elucidate the role of nonlocal *A-B* effect in the quantum-to-classical transition. We find that the nonlocal *A-B* effect is still alive in the classical limit of the matrix elements in quantum mechanics although such effect has no dynamical significance in classical mechanics. It states a fact that, under the correspondence principle of Heisenberg, the classical action J_k is a fractional number multiple of $\hbar$, while the physical system contains the *A-B* effect. There are two benefits according to the conclusion. First, the semiclassical quantization rule can be derived by the classical limit of matrix elements in the light of Heisenberg principle. Second, the conclusion can help to understand the role and position of topological effect in decoherence process and the connection between the quantum and classical mechanics.

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