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# Globally exponential stability condition of a class of neural networks with time-varying delays

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## Abstract

In this Letter, the globally exponential stability for a class of neural networks including Hopfield neural networks and cellular neural networks with time-varying delays is investigated. Based on the Lyapunov stability method, a novel and less conservative exponential stability condition is derived. The condition is delay-dependent and easily applied only by checking the Hamiltonian matrix with no eigenvalues on the imaginary axis instead of directly solving an algebraic Riccati equation. Furthermore, the exponential stability degree is more easily assigned than those reported in the literature. Some examples are given to demonstrate validity and excellence of the presented stability condition herein.

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## 1. Introduction

Both Hopfield neural networks (HNN) and cellular neural networks (CNN) have been studied extensively over the recent decades [1–5], and have been widely applied within various engineering and scientific fields such as neuro-biology, population biology, and computing technology. When neural networks are applied to signal processing systems or to the solution of nonlinear algebraic equations, it is essential to determine the existence of a unique equilibrium point and to establish its qualitative properties of stability. Although neural networks can be implemented by very large-scale integrated (VLSI) electronic circuits, the finite switching speed of amplifiers and the

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inherent communication time of neurons inevitably induce time-varying delays in the interaction between the neurons [4,5], and this may result in an oscillation phenomenon or network instability. In recent years, several researchers have developed delay-independent and delay-dependent stability criteria for a class of delayed neural networks by combining Razumikhin techniques with the Lyapunov functionals, the M-matrix, linear matrix inequality (LMI) formulation or the Lyapunov functions [4–19]. Usually, the delay-independent criterion may be overly restrictive when the delays are comparatively small. Therefore, a delay-dependent stability criterion determining the influence of time-varying delays upon the stability of neural networks is a fundamental concern.

This Letter considers a class of neural networks including Hopfield neural networks and cellular neural networks with time-varying delays, which can be described as follows:

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} f_j(x_j(t - \tau_j(t))) + J_i, \quad i = 1, \dots, n, \quad (1)$$

where  $n \geq 2$  is the number of neurons in the network,  $x_i$  denotes the state variable associated with the  $i$ th neuron, and  $c_i x_i(t)$  is an appropriately behaved function such that the solution of the model given in (1) remains bounded. The feedback matrix  $A = (a_{ij})_{n \times n}$  indicates the strength of the neuron interconnections within the network, while the delayed feedback matrix,  $B = (b_{ij})_{n \times n}$ , indicates the strength of the neuron interconnections within the network with time-varying delay  $\tau_j(t)$ . In addition, the activation function  $f_i$  is smooth and monotone in nature. It describes the manner in which the neurons respond to each other. Moreover,  $f_i$  satisfies  $0 < f'_i \leq \bar{M}_i$ ,  $i = 1, \dots, n$ , for Hopfield neural networks and  $f_i(x) = (|x + 1| - |x - 1|)/2$  for cellular neural networks, respectively. In (1),  $\tau_j(t) \geq 0$  represents time-varying delay parameter, and it is assumed that  $\tau_j^* = \max(\tau_j(t))$  and  $r_j^* = \max(\dot{\tau}_j(t)) < 1$  for  $1 \leq j \leq n$  and  $t \geq 0$ . Finally,  $J_i$  is the external constant input from outside of the system. The initial conditions of (1) are given by  $x_i(t) = \phi_i(t) \in C([- \tau_j^*, 0], \mathfrak{R})$ , where  $C([- \tau_j^*, 0], \mathfrak{R})$  denotes the set of all continuous functions from  $[- \tau_j^*, 0]$  to  $\mathfrak{R}$ .

The purpose of this Letter is to establish a novel and less conservative sufficient delay-dependent condition for the globally exponential stability of Hopfield neural networks and cellular neural networks with time-varying delays. It is noted that the stability degree can be easily solved. Moreover, the condition is compared with the previous results derived in the literature.

## 2. Preliminaries

The following notations are used throughout this Letter.  $M^T$  and  $\lambda(M)$  denote the transpose and the eigenvalue of a square matrix  $M$ , respectively. Furthermore, for  $x \in \mathfrak{R}^n$ , let  $\|x\| = (x^T x)^{1/2}$  denote the Euclidean vector norm, and for a matrix  $A \in \mathfrak{R}^{n \times n}$ , let  $\|A\|$  represent the norm of  $A$  induced by the Euclidean vector norm, i.e.,  $\|A\| = (\lambda_{\max}(A^T A))^{1/2}$ , where  $\lambda_{\max}(B)$  represents the maximum eigenvalue of matrix  $B$ .

It is assumed that (1) has an equilibrium point given as  $x^* = [x_1^*, x_2^*, \dots, x_n^*]^T$ . If  $z(t)$  is defined as  $z(t) = x(t) - x^*$ , where  $x = [x_1, x_2, \dots, x_n]^T$  and  $z = [z_1, z_2, \dots, z_n]^T$ , respectively, then (1) can be reexpressed as the following form:

$$\dot{z}_i(t) = -c_i z_i + \sum_{j=1}^n a_{ij} \phi_j(z_j(t)) + \sum_{j=1}^n b_{ij} \phi_j(z_j(t - \tau_j(t))), \quad i = 1, \dots, n, \quad (2)$$

where

$$\phi_j(z_j(t)) = f_j(z_j + x_j^*) - f_j(x_j^*), \quad \phi_j(z_j(t - \tau_j(t))) = f_j(z_j(t - \tau_j(t)) + x_j^*) - f_j(x_j^*).$$

Clearly, the origin,  $z = 0$ , is an equilibrium point of the system (2). In order to confirm that the origin of (2) is globally exponentially stable, let  $\hat{z}_i(t) = e^{\alpha t} z_i(t)$  and transform (2) into the following form:

$$\dot{\hat{z}}_i(t) = -(c_i - \alpha)\hat{z}_i + \sum_{j=1}^n a_{ij}\hat{\phi}_j(\hat{z}_j(t)) + \sum_{j=1}^n b_{ij}\tilde{\phi}_j(\hat{z}_j(t - \tau_j(t))), \quad i = 1, \dots, n, \quad (3)$$

where

$$\hat{\phi}_j(\hat{z}_j(t)) = e^{\alpha t} \phi_j(z_j(t)), \quad \tilde{\phi}_j(\hat{z}_j(t - \tau_j(t))) = e^{\alpha t} \phi_j(z_j(t - \tau_j(t))).$$

To establish the exponential stability of the model given in (1) or (2), it is first necessary to make the usual assumption regarding the activation functions  $f_i$  and definition for the exponential stability as follows.

**Assumption (H).** Each function  $f_i : \mathfrak{R} \rightarrow \mathfrak{R}$ ,  $i \in \{1, 2, \dots, n\}$  is bounded, and satisfies the Lipschitz condition with a Lipschitz constant  $L_i > 0$ , i.e.,  $|f_i(u) - f_i(v)| \leq L_i|u - v|$  for all  $u, v \in \mathfrak{R}$ .

**Definition 1** [14,20]. An equilibrium point,  $z = 0$ , is said to be exponentially stable if there exist  $\alpha > 0$  and  $\beta(\alpha) > 1$  such that  $\|z(t)\| \leq \beta(\alpha)e^{-\alpha t}(\sup_{-\tau \leq s \leq 0} \|z(s)\|)$ ,  $t \geq 0$ . Furthermore, the constant  $\alpha$  is called the degree of exponential stability.

**Remark 1.** An equilibrium point  $z = 0$  is asymptotically stable if the following conditions are satisfied:

- (i) the degree of exponential stability is zero, i.e.,  $\alpha = 0$  and  $\beta(0) = M > 1$  such that

$$\|z(t)\| \leq M \left( \sup_{-\tau \leq s \leq 0} \|z(s)\| \right), \quad t \geq 0;$$

- (ii)  $z(t)$  approaches zero as  $t$  approaches infinity, i.e.,  $\lim_{t \rightarrow \infty} z(t) = 0$ .

### 3. Main results

In this section, we present a sufficient globally exponential stability condition for the uniqueness of the equilibrium point  $x^*$  of system (1). The following lemmas are necessary for determining the exponential stability of (1):

**Lemma 1.** Define a  $2n \times 2n$  Hamiltonian matrix

$$H = \begin{bmatrix} -\hat{C} & DD^T \\ -(K_1 + K_2) - \varepsilon I_n & \hat{C} \end{bmatrix},$$

where  $\varepsilon$  is sufficiently small and positive constant,  $I_n$  is a  $n \times n$  identity matrix

$$\hat{C} = \text{diag}\{c_i, -\alpha\}, \quad i = 1, 2, \dots, n, \quad D = [A \quad B],$$

$$K_1 = \text{diag}\{L_j^2\}, \quad K_2 = \text{diag}\left\{\frac{e^{2\alpha\tau_j^*} L_j^2}{1 - r_j^*}\right\}, \quad j = 1, 2, \dots, n.$$

If  $-\hat{C}$  is a stable matrix and Hamiltonian matrix  $H$  has no eigenvalues on the imaginary axis, then the following algebraic Riccati equation (ARE):

$$-\hat{C}^T P - P\hat{C} + PDD^T P + (K_1 + K_2) + \varepsilon I_n = 0 \quad (4)$$

has a symmetric and positive definite solution  $P$  for a given  $\alpha > 0$ .

**Remark 2.** The proof is an immediate consequence of the Lemma 4 in the work of Doyle et al. [21], and it is omitted here.

**Lemma 2** [22]. For  $\hat{z}(t) = e^{\alpha t} z(t)$  and  $z(t)$  is the solution of system (2). If the origin of  $\hat{z}(t)$  is asymptotically stable, then the equilibrium point  $z = 0$  of the system (2) is exponentially stable with a stability degree  $\alpha$ .

With two lemmas given above, the main result of this Letter is stated in the following theorem.

**Theorem 1.** For the class of neural networks with time-varying delays defined in system (1) satisfying Assumption (H),  $\tau_j^* = \max(\tau_j(t))$  and  $r_j^* = \max(\dot{\tau}_j(t)) < 1$ . If  $-\hat{C}$  is a stable matrix and Hamiltonian matrix  $H$  defined in Lemma 1 for a given  $\alpha > 0$  has no eigenvalues on the imaginary axis, then the equilibrium point  $x^*$  of the system (1) is globally exponential stable with a stability degree  $\alpha$ .

**Proof.** We prove the theorem in the following three steps.

*Step 1.* We firstly transform (3) into a compact form as follows:

$$\dot{\hat{z}}(t) = -\hat{C}\hat{z}(t) + A\hat{\phi}(\hat{z}(t)) + B\tilde{\phi}(\hat{z}(t - \tau(t))), \quad (5)$$

where

$$\hat{z}(t) = [\hat{z}_1(t) \quad \hat{z}_2(t) \quad \dots \quad \hat{z}_n(t)]^T \in \mathbb{R}^n, \quad \hat{C} = \text{diag}\{c_i - \alpha\}, \quad i = 1, 2, \dots, n,$$

$$\hat{\phi}(\hat{z}(t)) = [\hat{\phi}_1(\hat{z}_1(t)) \quad \hat{\phi}_2(\hat{z}_2(t)) \quad \dots \quad \hat{\phi}_n(\hat{z}_n(t))]^T \in \mathbb{R}^n,$$

$$\tilde{\phi}(\hat{z}(t - \tau(t))) = [\tilde{\phi}_1(\hat{z}_1(t - \tau_1(t))) \quad \tilde{\phi}_2(\hat{z}_2(t - \tau_2(t))) \quad \dots \quad \tilde{\phi}_n(\hat{z}_n(t - \tau_n(t)))]^T \in \mathbb{R}^n.$$

*Step 2.* Since  $-\hat{C}$  is stable and the Hamiltonian matrix  $H$  has no eigenvalues on the imaginary axis. According to Lemma 1, the algebraic Riccati equation (ARE) as given in (4) has a symmetric and positive definite solution  $P$ . In order to confirm that the origin of (5) is globally asymptotically stable, a continuous Lyapunov function  $V$  is defined as

$$V(t) = \hat{z}(t)^T P \hat{z}(t) + \sum_{j=1}^n \frac{e^{2\alpha\tau_j^*}}{1 - r_j^*} \int_{t-\tau_j(t)}^t \hat{\phi}_j^2(\hat{z}_j(s)) ds. \quad (6)$$

It can easily be verified that  $V(t)$  is a non-negative function over  $[-\tau, +\infty)$  and that it is radially unbounded, i.e.,  $V(t) \rightarrow +\infty$  as  $\hat{z}(t) \rightarrow +\infty$ . Using the definition of  $\phi_j(z_j(t))$ ,  $\hat{\phi}_j(\hat{z}_j(t))$  and the Assumption (H) yields:

$$|\phi_j(z_j(t))| \leq L_j |z_j(t)|, \quad (7)$$

$$|\hat{\phi}_j(\hat{z}_j(t))| = |e^{\alpha t} \phi_j(z_j(t))| \leq L_j |e^{\alpha t} z_j(t)| = L_j |\hat{z}_j(t)|. \quad (8)$$

*Step 3.* Evaluating the time derivative of  $V$  along the trajectory of (5) gives:

$$\begin{aligned} \dot{V}(t) &= \dot{\hat{z}}(t)^T P \hat{z}(t) + \hat{z}(t)^T P \dot{\hat{z}}(t) + \sum_{j=1}^n \frac{e^{2\alpha\tau_j^*}}{1 - r_j^*} [\hat{\phi}_j^2(\hat{z}_j(t)) - (1 - \dot{\tau}_j(t)) \hat{\phi}_j^2(\hat{z}_j(t - \tau_j(t)))] \\ &= \hat{z}(t)^T (-\hat{C}^T P - P \hat{C}) \hat{z}(t) + \hat{z}(t)^T P (A \hat{\phi}(\hat{z}(t)) + B \tilde{\phi}(\hat{z}(t - \tau(t)))) \\ &\quad + (\hat{\phi}(\hat{z}(t))^T A^T + \tilde{\phi}(\hat{z}(t - \tau(t)))^T B^T) P \hat{z}(t) \end{aligned}$$

$$+ \sum_{j=1}^n \frac{e^{2\alpha\tau_j^*}}{1-r_j^*} \hat{\phi}_j^2(\hat{z}_j(t)) - \sum_{j=1}^n \frac{e^{2\alpha\tau_j^*}}{1-r_j^*} (1-\dot{\tau}_j(t)) \hat{\phi}_j^2(\hat{z}_j(t-\tau_j(t))). \quad (9)$$

Using the fact  $X^T Y + Y^T X \leq X^T X + Y^T Y$  for any matrices  $X$  and  $Y$  with appropriate dimensions obtains

$$\begin{aligned} \hat{z}(t)^T P A \hat{\phi}(\hat{z}(t)) + \hat{\phi}(\hat{z}(t))^T A^T P \hat{z}(t) &\leq \hat{z}(t)^T P A A^T P \hat{z}(t) + \hat{\phi}(\hat{z}(t))^T \hat{\phi}(\hat{z}(t)) \\ &\leq \hat{z}(t)^T P A A^T P \hat{z}(t) + \sum_{j=1}^n L_j^2 |\hat{z}_j(t)|^2 \\ &= \hat{z}(t)^T P A A^T P \hat{z}(t) + \hat{z}(t)^T K_1 \hat{z}(t), \end{aligned} \quad (10)$$

$$\begin{aligned} &\hat{z}(t)^T P B \tilde{\phi}(\hat{z}(t-\tau)) + \tilde{\phi}(\hat{z}(t-\tau))^T B^T P \hat{z}(t) \\ &\leq \hat{z}(t)^T P B B^T P \hat{z}(t) + \tilde{\phi}(\hat{z}(t-\tau))^T \tilde{\phi}(\hat{z}(t-\tau)) \\ &= \hat{z}(t)^T P B B^T P \hat{z}(t) + \sum_{j=1}^n e^{2\alpha t} \phi_j^2(z_j(t-\tau_j(t))) \\ &= \hat{z}(t)^T P B B^T P \hat{z}(t) + \sum_{j=1}^n e^{2\alpha\tau_j(t)} e^{2\alpha(t-\tau_j(t))} \phi_j^2(z_j(t-\tau_j(t))) \\ &= \hat{z}(t)^T P B B^T P \hat{z}(t) + \sum_{j=1}^n e^{2\alpha\tau_j(t)} \hat{\phi}_j^2(\hat{z}_j(t-\tau_j(t))) \\ &\leq \hat{z}(t)^T P B B^T P \hat{z}(t) + \sum_{j=1}^n e^{2\alpha\tau_j^*} \hat{\phi}_j^2(\hat{z}_j(t-\tau_j(t))). \end{aligned} \quad (11)$$

The last two terms in (9) can be further derived as follows:

$$\sum_{j=1}^n \frac{e^{2\alpha\tau_j^*}}{1-r_j^*} \hat{\phi}_j^2(\hat{z}_j(t)) \leq \sum_{j=1}^n \frac{e^{2\alpha\tau_j^*}}{1-r_j^*} L_j^2 \hat{z}_j^2(t) = \hat{z}^T K_2 \hat{z}, \quad (12)$$

$$- \sum_{j=1}^n \frac{1-\dot{\tau}_j(t)}{1-r_j^*} e^{2\alpha\tau_j^*} \hat{\phi}_j^2(\hat{z}_j(t-\tau_j(t))) \leq - \sum_{j=1}^n e^{2\alpha\tau_j^*} \hat{\phi}_j^2(\hat{z}_j(t-\tau_j(t))). \quad (13)$$

By applying (10)–(13) to (9), we obtain

$$\begin{aligned} \dot{V}(t) &\leq \hat{z}(t)^T [-\hat{C}^T P - P \hat{C} + P(AA^T + BB^T)P + (K_1 + K_2)] \hat{z}(t) \\ &= \hat{z}(t)^T [-\hat{C}^T P - P \hat{C} + PDD^T P + (K_1 + K_2)] \hat{z}(t) \\ &= -\varepsilon \hat{z}(t)^T \hat{z}(t). \end{aligned} \quad (14)$$

It can be shown that  $\dot{V}(t) \leq -\varepsilon \|\hat{z}(t)\|^2$  holds. According to Lyapunov theory, we conclude that  $V(t)$  converges to zero asymptotically and thereby implying that the equilibrium point  $\hat{z} = 0$  is globally asymptotically stable. Furthermore, by applying Lemma 2, we can conclude that  $z(t)$  converges to zero globally and exponentially with a convergence rate of  $\alpha$ , namely,  $\|z(t)\| \leq \beta(\alpha) e^{-\alpha t} (\sup_{-\tau \leq s \leq 0} \|z(s)\|)$  is satisfied. Therefore, every trajectory  $x(t)$  of (1) must converge exponentially toward the equilibrium point  $x^*$ , with a convergence rate of  $\alpha$ . This completes the proof.  $\square$

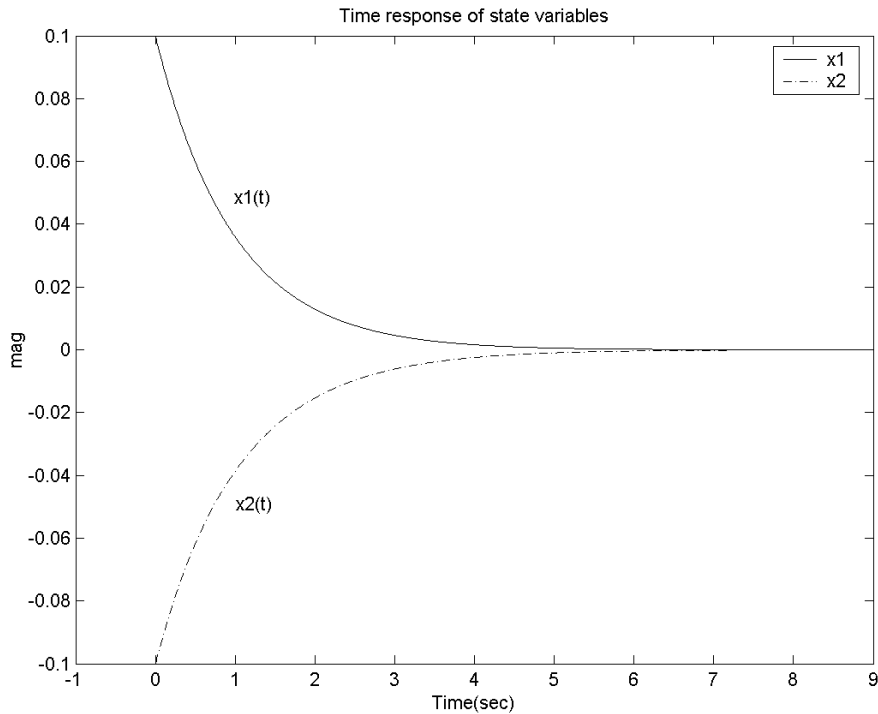


Fig. 1. Time responses of state variables for Example 1.

#### 4. Illustrative examples

The sufficient condition for globally exponential stability presented in this Letter is less restrictive than some of those presented in the published literature. Additionally, the stability degree is easily assigned. The relative advantages of the proposed condition are presented by means of the following three examples.

**Example 1.** Consider a delayed neural network presented by Yu et al. [13] as follows:

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^2 a_{ij} f_j(x_j(t)) + \sum_{j=1}^2 b_{ij} f_j(x_j(t - \tau_j(t))), \quad i = 1, 2. \quad (15)$$

The activation functions are given by  $f_i(x) = 0.05(|x + 1| - |x - 1|)$ ,  $i = 1, 2$ , therefore, the Lipschitz constants are  $L_1 = L_2 = 0.1$ . For  $A = \begin{bmatrix} -0.1 & 0.3 \\ 0 & -0.1 \end{bmatrix}$  and  $B = \begin{bmatrix} 0.5 & 0.4 \\ 0.15 & 0.5 \end{bmatrix}$ , it has been reported in [13] that the system has a unique and globally asymptotically stable equilibrium point for  $0 \leq \tau_1(t), \tau_2(t) \leq 0.3623$ . However, herein, if we specify the time-varying delays as follows:  $\tau_1(t) = \tau_2(t) = (1 - e^{-t})/(1 + e^{-t})$ , which satisfy  $0 \leq \tau_j(t) \leq 1 = \tau_j^*$ ,  $0 \leq \dot{\tau}_j(t) \leq 1/2 = r_j^*$ ,  $j = 1, 2$ . The delays given here have a broader interval and include that in [13]. Then we specify  $\varepsilon = 10^{-5}$ ,  $K_1 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$  and  $K_2 = \frac{2\alpha\tau_j^*}{1-r_j^*} \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$ , it is easily found that the matrix  $-\hat{C} = \begin{bmatrix} -1+\alpha & 0 \\ 0 & -1+\alpha \end{bmatrix}$  is stable and Hamiltonian matrix  $H$  has no eigenvalues on the imaginary axis for the given  $\alpha = 0.7437$ . Thus according to Theorem 1, the system (15) is globally exponential stable with a stability degree  $\alpha = 0.7437$  at least. Fig. 1 depicts the time responses of the state variables  $x_1(t)$  and  $x_2(t)$  for the initial condition  $[x_1(s) \ x_2(s)] = [0.1 \ -0.1]$ , where  $-1 \leq s \leq 0$ . Similarly, for matrices  $A = \begin{bmatrix} -0.25 & 0.3 \\ 0.15 & -0.25 \end{bmatrix}$  and  $B = \begin{bmatrix} 0.25 & 0.3 \\ 0.15 & 0.25 \end{bmatrix}$ , the system has a unique and globally asymptotically stable equilibrium point for  $0 \leq \tau_1(t), \tau_2(t) \leq 1.5947$  [13]. Also, if we specify

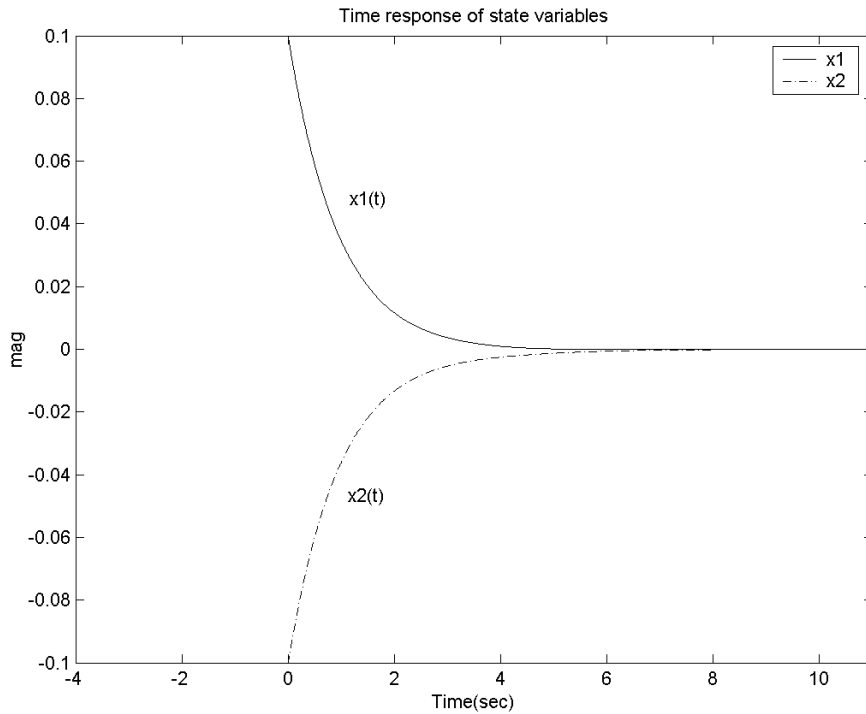


Fig. 2. Time responses of state variables for Example 1.

$\alpha = 0.3958$ ,  $\varepsilon = 10^{-4}$  and  $0 \leq \tau_1(t) = \tau_2(t) = 4(1 - e^{-0.2t}) \leq 4$ , the result of this Letter still guarantees that the system (15) is globally exponential stable with a stability degree. Fig. 2 depicts the time responses of the state variables  $x_1(t)$  and  $x_2(t)$  for the initial condition  $[x_1(s) \ x_2(s)] = [0.1 \ -0.1]$ , where  $-4 \leq s \leq 0$ .

**Example 2.** For a cellular neural network with time-varying delays as given in (1) and the activation function  $f_i(x) = 0.5(|x + 1| - |x - 1|)$ , it has been reported by Zhou and Cao [4] that the system is exponentially stable if one of the following two conditions is satisfied:

$$(i) \quad \min_{1 \leq i \leq n} \left( c_i - L_i \sum_{j=1}^n |a_{ji}| \right) > \max_{1 \leq i \leq n} \left( L_i \sum_{j=1}^n |b_{ji}| \right) > 0, \quad (16)$$

$$(ii) \quad \min_{1 \leq i \leq n} \left( 2c_i - \sum_{j=1}^n [L_i(|a_{ij}| + |b_{ij}|) + L_j|a_{ji}|] \right) > \max_{1 \leq i \leq n} \left( L_i \sum_{j=1}^n |b_{ji}| \right) > 0. \quad (17)$$

As a comparative example, the system's parameters are given as follows:

$$A = (a_{ij})_{2 \times 2} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad B = (b_{ij})_{2 \times 2} = \begin{bmatrix} b_{11} & -0.8 \\ b_{21} & 0.15 \end{bmatrix}, \quad c_1 = c_2 = 4, \quad J_1 = J_2 = 1,$$

and delays  $0 \leq \tau_j(t) = 1 - e^{-0.3t} \leq \tau_j^* = 1$ . The Lipschitz constants of the activation functions are  $L_1 = L_2 = 1$ . Fig. 3 shows the stability region in the  $b_{11}$ – $b_{21}$  parameter space of the proposed condition with  $\alpha = 10^{-4}$ ,  $\varepsilon = 10^{-7}$  and that of [4]. It is noted that Fig. 3 demonstrates that the results of the proposed condition are less restrictive than those given in [4]. More precisely, if the parameters  $b_{11} = b_{21} = 1.2$  are given, then the conditions of [4] given in Eqs. (16) and (17) are not applicable. However, Theorem 1 proposed in this Letter can be applied to ascertain that

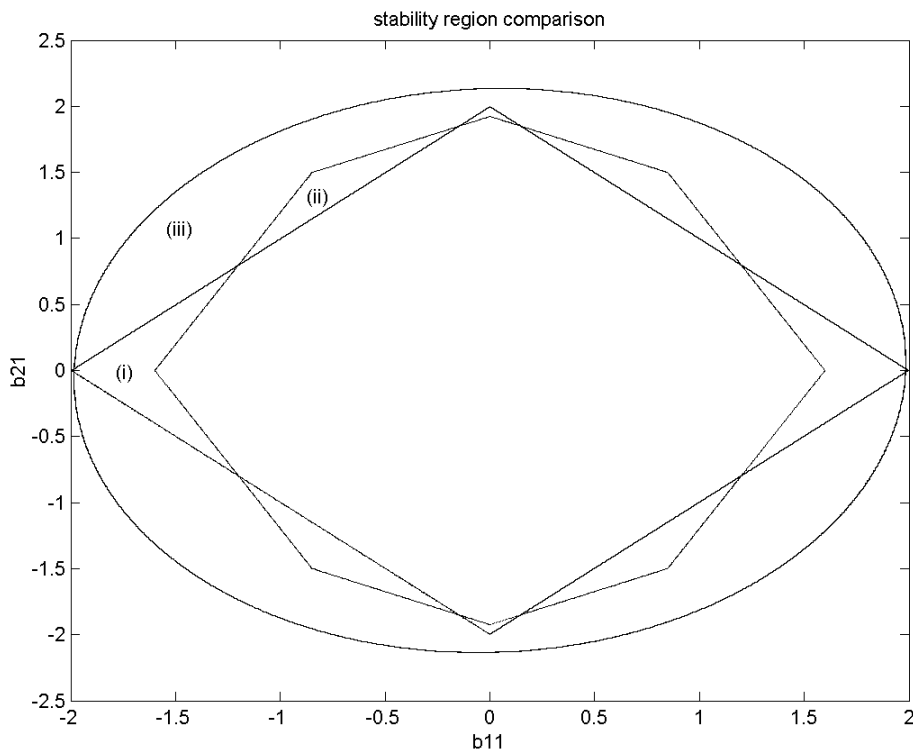


Fig. 3. The stability region in the  $b_{11}$ - $b_{21}$  parameter space determined by (i) Theorem 1 of [4], (ii) Theorem 2 of [4] and (iii) the proposed herein.

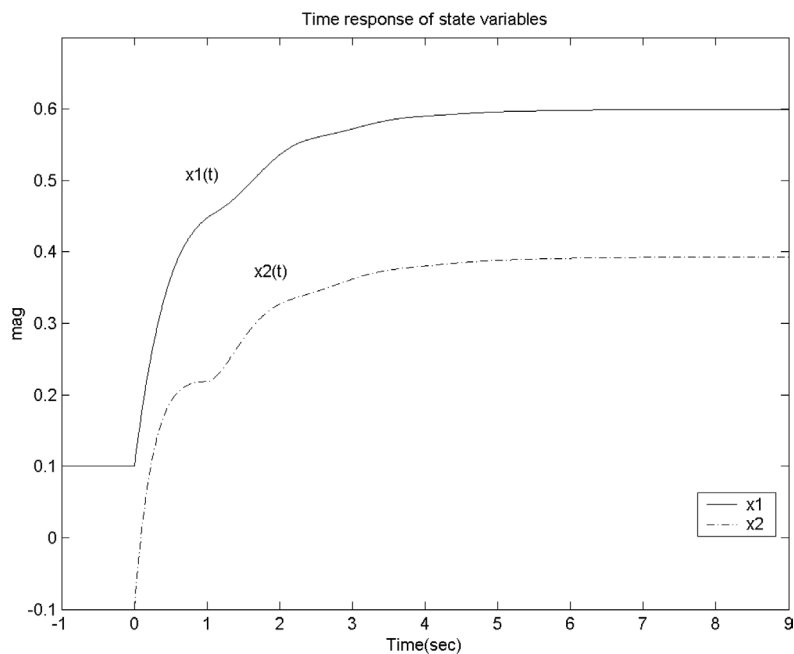


Fig. 4. Time responses of state variables for Example 2.

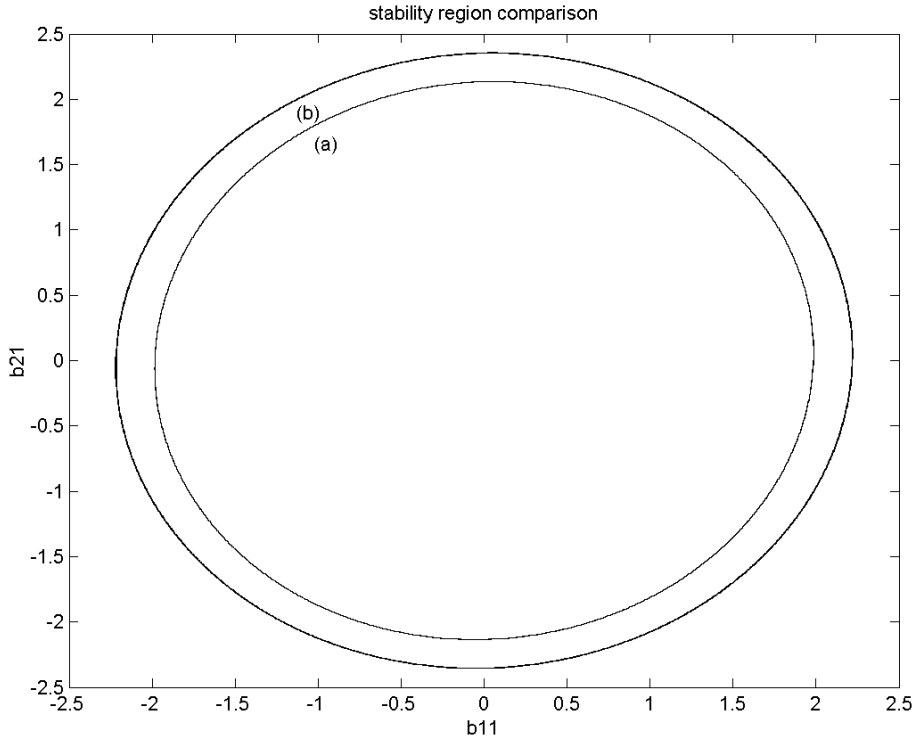


Fig. 5. The stability region in the  $b_{11}$ – $b_{21}$  parameter space for two different delays (a)  $0 \leq \tau_i(t) = 1 - e^{-0.3t} \leq 1$  and (b)  $0 \leq \tau_i(t) = 1 - e^{-0.1t} \leq 1$ .

the system has a unique equilibrium point. Fig. 4 depicts the time responses of the state variables  $x_1(t)$  and  $x_2(t)$  with the unique equilibrium point  $x^* = [0.59922 \ 0.39286]^T$  for the initial condition  $[x_1(s) \ x_2(s)] = [0.1 \ -0.1]$ , where  $-1 \leq s \leq 0$  and  $b_{11} = b_{21} = 1.2$ . Fig. 5 shows the stability region in the  $b_{11}$ – $b_{21}$  parameter space of the proposed condition for two different time-varying delays  $0 \leq \tau_1(t) = \tau_2(t) = 1 - e^{-0.3t} \leq 1$  and  $0 \leq \tau_1(t) = \tau_2(t) = 1 - e^{-0.1t} \leq 1$ , that also indicates that the proposed stability condition is delay-dependent as well as affected by the derivative of delays.

**Example 3.** To compare the degree of exponential stability for the cellular neural networks with time-varying delays (DCNN) presented by Hwang et al. [19] and that proposed herein, the system's parameters are given as

$$J_1 = J_2 = 1, \quad c_1 = c_2 = 4, \quad L_1 = L_2 = 1,$$

$$A = \begin{bmatrix} 0.7 & -0.8 \\ 0.6 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 & -0.6 \\ -0.4 & 0.4 \end{bmatrix}.$$

The degree of exponential stability for the system estimated by [19] is 0.6851 for  $0 \leq \tau_j(t) \leq 1$ ,  $j = 1, 2$ . By choosing  $\varepsilon = 10^{-7}$ , Theorem 1 proposed herein ascertains that the system is exponentially stable with a convergence rate of  $\alpha = 0.7764$  for  $\tau_j(t) = 1 - e^{-0.001t}$ , i.e.  $\tau_j(t) \leq 1$ ,  $\dot{\tau}_j(t) \leq 0.001$  for  $j = 1, 2$ . The speed of convergence of the system derived in this Letter is faster than that estimated by [19].

## 5. Conclusions

This Letter has presented a new sufficient condition to guarantee the globally exponential stability for a class of neural networks including Hopfield neural networks and cellular neural networks with time-varying delays. The proposed condition can be easily verified by checking the Hamiltonian matrix with no eigenvalues on the imaginary axis instead of directly solving an algebraic Riccati equation. Furthermore, the stability degree is easily solved. The results have indicated that the proposed stability condition is delay-dependent and even affected by the derivative of delays, and that it is less restrictive than some other conditions presented within the published literature.

## References

- [1] L.O. Chua, L. Yang, *IEEE Trans. Circuits Systems* 35 (1988) 1257.
- [2] J.J. Hopfield, *Proc. Natl. Acad. Sci. USA* 79 (1982) 2554.
- [3] D.G. Kelly, *IEEE Trans. Biomed. Eng.* 37 (1990) 231.
- [4] D. Zhou, J. Cao, *Appl. Math. Comput.* 131 (2002) 487.
- [5] J. Cao, *IEEE Trans. Circuits Systems* 48 (2001) 1330.
- [6] J.H. Li, A.N. Michel, *IEEE Trans. Circuits Systems* 35 (1988) 976.
- [7] S. Arik, *IEEE Trans. Circuits Systems* 47 (2000) 1089.
- [8] M. Joy, *J. Math. Anal. Appl.* 232 (1999) 61.
- [9] Y.J. Cao, Q.H. Wu, *IEEE Trans. Neural Networks* 7 (1996) 1533.
- [10] P. Baldi, A.F. Atiya, *IEEE Trans. Neural Networks* 5 (1994) 612.
- [11] S. Mohamad, K. Gopalsamy, *Appl. Math. Comput.* 135 (2002) 17.
- [12] J. Cao, *Phys. Rev. E* 60 (1999) 3244.
- [13] G.J. Yu, C.Y. Lu, S.H. Tsai, T.J. Su, B.D. Liu, *IEEE Trans. Circuits Systems* 50 (5) (2003) 677.
- [14] E. Yucel, S. Arik, *Physica D* 191 (2004) 314.
- [15] S. Guo, L. Huang, *Phys. Rev. E* 67 (2003) 061902.
- [16] D. Zhou, L. Zhang, J. Cao, *Appl. Math. Comput.* 151 (2004) 379.
- [17] X. Liao, G. Chen, E.N. Sanchez, *Neural Networks* 15 (2002) 855.
- [18] W.H. Chen, Z.H. Guan, X. Lu, *Phys. Lett. A* 326 (2004) 355.
- [19] C.C. Hwang, C.J. Cheng, T.L. Liao, *Phys. Lett. A* 319 (1–2) (2003) 157.
- [20] H.K. Khalil, *Nonlinear Systems*, Macmillan, New York, 1992.
- [21] J.C. Doyle, K. Glover, P.P. Khargonekar, B.A. Francis, *IEEE Trans. Automatic Control* 34 (1989) 831.
- [22] K.K. Shyu, J.J. Yan, *Int. J. Control* 57 (1) (1993) 237.