Exponential Synchronization of a Class of Neural Networks With Time-Varying Delays

Chao-Jung Cheng, Teh-Lu Liao, Jun-Juh Yan, and Chi-Chuan Hwang

***Abstract*—This paper aims to present a synchronization scheme for a class of delayed neural networks, which covers the Hopfield neural networks and cellular neural networks with time-varying delays. A feedback control gain matrix is derived to achieve the exponential synchronization of the drive-response structure of neural networks by using the Lyapunov stability theory, and its exponential synchronization condition can be verified if a certain Hamiltonian matrix with no eigenvalues on the imaginary axis. This condition can avoid solving an algebraic Riccati equation. Both the cellular neural networks and Hopfield neural networks with time-varying delays are given as examples for illustration.**

***Index Terms*—Chaotic systems, Hamiltonian matrix, neural networks, synchronization.**

I. Introduction

Both Hopfield neural networks and cellular neural networks have become a field of active research over the past two decades for their potential applications in modeling complex dynamics [1]–[3]. Both them have been successfully applied in solving various linear and nonlinear programming problems, as well as in the applications of image processing. However, stability analysis in this kind of neural networks is a very important issue, and several stability criteria have been developed in the literature [3], [4] and references cited therein.

On the other hand, it has been well known that a chaotic system is a nonlinear deterministic system with complex and unpredictable behavior. Furthermore, chaotic behaviors produced by this kind of neural networks have also been found and investigated in [5]–[7]. The synchronization of chaotic systems has been extensively studied over the past two decades due to its potential applications in creating secure communication systems [8]–[15]. After the drive-response concept introduced by Pecora and Carroll in their pioneering work [8], several different approaches including some conventional linear control techniques and advanced nonlinear control schemes to achieve synchronization of the chaotic systems or neural networks have been proposed in the literature [16]–[20]. Recently, a synchronization criterion for coupled delayed neural networks based on the Lyapunov functional method and Hermitian matrices theory is derived in [21]. However, the above synchronization schemes are derived for chaotic systems or for a class of neural networks with or without constant time delays.

In this paper, a synchronization scheme for a class of neural networks with time-varying delays is proposed. Based on the Lyapunov stability theory and drive-response synchronization concept, a control law with an appropriate gain matrix is derived to achieve synchronization of the drive-response-based neural networks with time-varying delays. The elements of the gain matrix are easily determined by checking a certain Hamiltonian matrix if its eigenvalues lie on the imaginary axis or not instead of arduously solving an algebraic Riccati equation.

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C.-J. Cheng is with the Department of Information Engineering, Kun Shan University, Tainan 710, Taiwan, R.O.C.

T.-L. Liao and C.-C. Hwang are with the Department of Engineering Science, National Cheng Kung University, Tainan 701, Taiwan, R.O.C. (e-mail: tlliao@mail.ncku.edu.tw).

J.-J. Yan is with the Department of Computer and Communication, Shu-Te University, Kaohsiung 824, Taiwan, R.O.C.

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II. Synchronization Problem Formulation

Based on the drive-response concept, the unidirectional coupled neural networks are described by the following equations:

$$\begin{aligned}\dot{x}_i(t) =& -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\left(x_j(t)\right)\\ &+ \sum_{j=1}^{n} b_{ij} f_j\left(x_j\left(t-\tau_j(t)\right)\right) + J_i,\\ & i = 1,\dots,n \end{aligned} \quad (1)$$

$$\begin{aligned}\dot{z}_i(t) =& -c_i z_i(t) + \sum_{j=1}^{n} a_{ij} f_j\left(z_j(t)\right)\\ &+ \sum_{j=1}^{n} b_{ij} f_j\left(z_j\left(t-\tau_j(t)\right)\right) + J_i + u_i(t),\\ & i = 1,\dots,n \end{aligned} \quad (2)$$

where $n \geq 2$ denotes the number of neurons in the network, x_i is the state variable associated with the ith neuron, and $c_i x_i(t)$ is an appropriately behaved function remaining the solution of drive neural networks (1) bounded. $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ indicate the interconnection strength among neurons without and with time-varying delay $\tau_j(t) \geq 0$, respectively. The function f_i describes the manner in which the neurons respond to each other, and $f_i(x) = \tanh(x)$ for Hopfield neural networks; and $f_i(x) = 0.5(|x+1| - |x-1|)$ for cellular neural networks, respectively. While J_i is an external constant input; and $u_i(t)$ is unidirectional coupled term which is considered as the control input and will be appropriately designed to obtain a certain control objective. Furthermore, it is assumed that $\tau_j^* = \max(\tau_j(t))$ and $r_j^* = \max(\dot{\tau}_j(t)) < 1$ for $1 \leq j \leq n$ and $t \geq 0$, and the systems (1) and (2) possess initial conditions $x_i(t) = \psi_i(t) \in C([-\tau_j^*, 0], \Re)$ and $z_i(t) = \varphi_i(t) \in C([-\tau_j^*, 0], \Re)$, respectively, where $C([-\tau_j^*, 0], \Re)$ denotes the set of all continuous functions from $[-\tau_j^*, 0]$ to $\Re$.

Before proceeding, an assumption regarding f_i and definition of exponential synchronization are given below.

(H) $f_i : \Re \to \Re$, $i \in \{1, 2, \dots, n\}$ is bounded, and satisfies the Lipschitz condition with a Lipschitz constant $L_i > 0$, i.e. $|f_i(u) - f_i(v)| \leq L_i |u - v|$ for all $u, v \in \Re$.

Definition 1 [22]: The system (1) and the uncontrolled system (2) [i.e., $u \equiv 0$ in (2)] are said to be exponentially synchronized if there exist constants $\beta(\alpha) \geq 1$ and $\alpha > 0$ such that $|x_i(t) - z_i(t)| \leq \beta(\alpha) \sup_{-\tau_j^* \leq s \leq 0} |x_i(s) - z_i(s)| \exp^{-\alpha t}$ for any $t \geq 0$. Constant α is said to be the degree of exponential synchronization.

The paper aims to determine the control input u_i associated with the state-feedback for the purpose of exponentially synchronizing the unidirectional coupled identical chaotic neural networks with the same system's parameters but the differences in initial conditions.

III. Main Results

Let the synchronization error $e(t)$ be defined as follows: $e(t) = [e_1(t), e_2(t), \cdots, e_i(t), \cdots, e_n(t)]^T$, where $e_i(t) = x_i(t) - z_i(t)$. Therefore, the error dynamics between (1) and (2) can be expressed by

$$\begin{aligned}\dot{e}_i(t) = -\Bigg(& c_i e_i(t) - \sum_{j=1}^{n} a_{ij}\left(f_j\left(e_j(t) + z_j(t)\right) - f_j\left(z_j(t)\right)\right)\\ & - \sum_{j=1}^{n} b_{ij}\left(f_j\left(e_j\left(t-\tau_j(t)\right) + z_j\left(t-\tau_j(t)\right)\right)\right.\end{aligned}$$

$$-f_j\left(z_j(t-\tau_j(t))\right))) - u_i(t),$$
$$i = 1, \ldots, n. \tag{3}$$

The control input associated with the state-feedback is designed as follows:

$$[u_1(t), u_2(t), \cdots, u_n(t)]^T = \Omega\,[e_1(t), e_2(t), \cdots, e_n(t)]^T = \Omega e \tag{4}$$

where $\Omega = (\omega_{ij})_{n\times n}$ is the gain matrix to be determined for synchronizing both a *drive system* and *response system*. Furthermore, if a new error $\hat{e}_i(t)$ is defined by $\hat{e}_i(t) = e^{\alpha t} e_i(t)$, then the dynamics of (3) can be transformed into the following form:

$$\begin{aligned}\dot{\hat{e}}_i(t) =& -(c_i - \alpha)\hat{e}_i(t) + \sum_{j=1}^{n} a_{ij}\hat{\phi}_j\left(\hat{e}_j(t)\right)\\ &+ \sum_{j=1}^{n} b_{ij}\widetilde{\phi}_j\left(\hat{e}_j\left(t-\tau_j(t)\right)\right) - \sum_{j=1}^{n}\omega_{ij}\hat{e}_j(t),\\ & i = 1, \ldots, n\end{aligned} \tag{5}$$

where

$$\begin{aligned}\hat{\phi}_j\left(\hat{e}_j(t)\right) &= e^{\alpha t}\phi_j\left(e_j(t)\right)\\ \widetilde{\phi}_j\left(\hat{e}_j\left(t-\tau_j(t)\right)\right) &= e^{\alpha t}\phi_j\left(e_j\left(t-\tau_j(t)\right)\right)\\ \phi_j\left(e_j(t)\right) &= f_j(e_j + z_j) - f_j(z_j);\\ \phi_j\left(e_j\left(t-\tau_j(t)\right)\right) &= f_j\left(e_j\left(t-\tau_j(t)\right) + z_j\left(t-\tau_j(t)\right)\right)\\ &\quad - f_j\left(z_j\left(t-\tau_j(t)\right)\right)\end{aligned}$$

For further deriving the exponential synchronization condition on the control law, the following Lemmas are needed.

Lemma 1: Define a $2n \times 2n$ Hamiltonian matrix $H = \begin{bmatrix} -\widetilde{C} & DD^T \\ -(K_1+K_2)-\varepsilon I_n & \widetilde{C}^T\end{bmatrix}$, where ε is sufficiently small and positive constant, I_n is a $n\times n$ identity matrix $\widetilde{C} = diag(c_i - \alpha) + \Omega$, $i = 1, 2\cdots, n$, $D = [A \quad B]$, $K_1 = diag(L_j^2)$ and $K_2 = diag(e^{2\alpha\tau_j^*}L_j^2/(1-r_j^*))$, $j = 1, 2\cdots, n$. If $-\widetilde{C}$ is a stable matrix and Hamiltonian matrix H has no eigenvalues on the imaginary axis, then the algebraic Riccati equation (ARE)

$$-\widetilde{C}^T P - P\widetilde{C} + PDD^T P + (K_1 + K_2) + \varepsilon I_n = 0 \tag{6}$$

has a symmetric and positive definite solution P for a given $\alpha > 0$.

Remark 1: The proof is an immediate consequence of the Lemma 4 in the work of Doyle *et al.* [23], and therefore it is omitted here.

Remark 2: A real matrix $-\widetilde{C}$ is stable if and only if all of its eigenvalues have negative real parts. All eigenvalues of $-\widetilde{C}$ defined in Lemma 1 can be arbitrarily assigned by appropriately choosing the controller gain matrix Ω. Especially, if we choose the gain matrix as a diagonal matrix $\Omega = diag(\omega_i)$ and $\omega_i > \alpha - c_i$, $i = 1, 2\cdots, n$, then the eigenvalues of $-\widetilde{C}$ are $-(\omega_i + c_i - \alpha) < 0$, $i = 1, 2\cdots, n$, which implies that $-widetildeC$ is a stable matrix.

Lemma 2 [22]: From the definition of $\hat{e}(t)$ and the solution $e(t)$ of system (3). If the origin of $\hat{e}(t)$ is asymptotically convergent, then $e(t)$ is exponentially convergent with a synchronization degree α.

Main Theorem: For these drive and response neural networks (1) and (2) which satisfy assumption (H), if the controller gain matrix Ω in (4) is suitably designed such that $-\widetilde{C}$ is a stable matrix and Hamiltonian matrix H defined in Lemma 1 for a given $\alpha > 0$ has no eigenvalues on the imaginary axis, then the networks (1) and (2) are synchronized exponentially with a degree α at least.

Proof:

Step 1: Transform (5) into a compact form as follows:

$$\dot{\hat{e}}(t) = -\widetilde{C}\hat{e}(t) + A\hat{\phi}\left(\hat{e}(t)\right) + B\widetilde{\phi}\left(\hat{e}\left(t-\tau(t)\right)\right) \tag{7}$$

where

$$\begin{aligned}\hat{e}(t) &= [\hat{e}_1(t)\ \hat{e}_2(t)\ \cdots\ \hat{e}_n(t)]^T \in \Re^n;\\ \widetilde{C} &= diag\{c_i - \alpha\} + \Omega, \quad i = 1, 2\cdots, n;\\ \hat{\phi}\left(\hat{e}(t)\right) &= \left[\hat{\phi}_1\left(\hat{e}_1(t)\right), \hat{\phi}_2\left(\hat{e}_2(t)\right)\ldots, \hat{\phi}_n\left(\hat{e}_n(t)\right)\right]^T\\ &\in R^n \text{ and}\\ \widetilde{\phi}\left(\hat{e}\left(t-\tau(t)\right)\right) &= \left[\widetilde{\phi}_1\left(\hat{e}_1\left(t-\tau_1(t)\right)\right), \widetilde{\phi}_2\left(\hat{e}_2\left(t-\tau_2(t)\right)\right)\right.\\ &\quad \left.\ldots, \widetilde{\phi}_n\left(\hat{e}_n\left(t-\tau_n(t)\right)\right)\right]^T \in R^n.\end{aligned}$$

Since $-\widetilde{C}$ is stable and the Hamiltonian matrix H has no eigenvalues on the imaginary axis. According to Lemma 1, the algebraic Riccati equation (ARE) in (6) has a symmetric and positive definite solution P. To confirm that the origin of (7) is globally asymptotically convergent, a continuous Lyapunov functional V is defined as follows:

$$V(t) = \hat{e}(t)^T P\hat{e}(t) + \sum_{j=1}^{n}\frac{e^{2\alpha\tau_j^*}}{1-r_j^*}\int_{t-\tau_j(t)}^{t}\hat{\phi}_j^2\left(\hat{e}_j(s)\right)ds. \tag{8}$$

It can be easily verified that $V(t)$ is a nonnegative function over $\lfloor -\tau^*, +\infty)$ and radially unbounded, i.e. $V(t) \to +\infty$ as $\hat{e}(t) \to +\infty$. Using the definition of $\phi_j(e_j(t))$, $\hat{\phi}_j(\hat{e}_j(t))$ and the assumption (H) yields

$$\left|\phi_j\left(e_j(t)\right)\right| \le L_j\left|e_j(t)\right| \tag{9}$$

and

$$\left|\hat{\phi}_j\left(\hat{e}_j(t)\right)\right| = \left|e^{\alpha t}\phi_j\left(e_j(t)\right)\right| \le L_j\left|e^{\alpha t}e_j(t)\right| = L_j\left|\hat{e}_j(t)\right|. \tag{10}$$

Step 2: Evaluating the time derivative of V along the trajectory of (7) gives

$$\begin{aligned}\dot{V}(t) =& \dot{\hat{e}}(t)^T P\hat{e}(t) + \hat{e}(t)^T P\dot{\hat{e}}(t) + \sum_{j=1}^{n}\frac{e^{2\alpha\tau_j^*}}{1-r_j^*}\\ &\times\left(\hat{\phi}_j^2\left(\hat{e}_j(t)\right) - \left(1-\dot{\tau}_j(t)\right)\hat{\phi}_j^2\left(\hat{e}_j\left(t-\tau_j(t)\right)\right)\right)\\ =& \hat{e}(t)^T(-\widetilde{C}^T P - P\widetilde{C})\hat{e}(t)\\ &+ \hat{e}(t)^T P\left(A\hat{\phi}\left(\hat{e}(t)\right) + B\widetilde{\phi}\left(\hat{e}(t-\tau)\right)\right)\\ &+ \left(\hat{\phi}\left(\hat{e}(t)\right)^T A^T + \widetilde{\phi}\left(\hat{e}(t-\tau)\right)^T B^T\right)P\hat{e}(t)\\ &+ \sum_{j=1}^{n}\frac{e^{2\alpha\tau_j^*}}{1-r_j^*}\left(\hat{\phi}_j^2\left(\hat{e}_j(t)\right)\right) - \sum_{j=1}^{n}\frac{e^{2\alpha\tau_j^*}}{1-r_j^*}\\ &\times\left(\left(1-\dot{\tau}_j(t)\right)\hat{\phi}_j^2\left(\hat{e}_j\left(t-\tau_j(t)\right)\right)\right).\end{aligned} \tag{11}$$

By using the fact $X^TY + Y^TX \le X^TX + Y^TY$ for any matrices X and Y with appropriate dimensions, we obtain

$$\begin{aligned}&\hat{e}(t)^T PA\hat{\phi}\left(\hat{e}(t)\right) + \hat{\phi}\left(\hat{e}(t)\right)^T A^T P\hat{e}(t)\\ &\quad\le \hat{e}(t)^T PAA^T P\hat{e}(t) + \hat{\phi}\left(\hat{e}(t)\right)^T\hat{\phi}\left(\hat{e}(t)\right)\\ &\quad\le \hat{e}(t)^T PAA^T P\hat{e}(t) + \sum_{j=1}^{n} L_j^2\left|\hat{e}_j(t)\right|^2\\ &\quad= \hat{e}(t)^T PAA^T P\hat{e}(t) + \hat{e}(t)^T K_1\hat{e}(t)\end{aligned} \tag{12}$$

and similarly

$$\begin{aligned}&\hat{e}(t)^T PB\widetilde{\phi}\left(\hat{e}(t-\tau)\right) + \widetilde{\phi}\left(\hat{e}(t-\tau)\right)^T B^T P\hat{e}(t)\\ &\quad\le \hat{e}(t)^T PBB^T P\hat{e}(t) + \widetilde{\phi}\left(\hat{e}(t-\tau)\right)^T\widetilde{\phi}\left(\hat{e}(t-\tau)\right)\end{aligned}$$

$$
\begin{aligned}
&= \hat{e}(t)^T PBB^T P\hat{e}(t) + \sum_{j=1}^{n} e^{2\alpha t}\phi_j^2\left(e_j\left(t-\tau_j(t)\right)\right)\\
&= \hat{e}(t)^T PBB^T P\hat{e}(t)\\
&\quad + \sum_{j=1}^{n} e^{2\alpha\tau_j(t)} e^{2\alpha\left(t-\tau_j(t)\right)}\phi_j^2\left(e_j\left(t-\tau_j(t)\right)\right)\\
&= \hat{e}(t)^T PBB^T P\hat{e}(t) + \sum_{j=1}^{n} e^{2\alpha\tau_j(t)}\hat{\phi}_j^2\left(\hat{e}_j\left(t-\tau_j(t)\right)\right)\\
&\le \hat{e}(t)^T PBB^T P\hat{e}(t)\\
&\quad + \sum_{j=1}^{n} e^{2\alpha\tau_j^*}\hat{\phi}_j^2\left(\hat{e}_j\left(t-\tau_j(t)\right)\right). \qquad (13)
\end{aligned}
$$

The last two terms in (11) can be further derived as follows:

$$
\sum_{j=1}^{n}\frac{e^{2\alpha\tau_j^*}}{1-r_j^*}\hat{\phi}_j^2\left(\hat{e}_j(t)\right) \le \sum_{j=1}^{n}\frac{e^{2\alpha\tau_j^*}}{1-r_j^*}L_j^2\hat{e}_j(t)^2 = \hat{e}^T K_2\hat{e} \qquad (14)
$$

and

$$
\begin{aligned}
&-\sum_{j=1}^{n}\frac{1-\dot{\tau}_j(t)}{1-r_j^*}e^{2\alpha\tau_j^*}\hat{\phi}_j^2\left(\hat{e}_j\left(t-\tau_j(t)\right)\right)\\
&\qquad\qquad \le -\sum_{j=1}^{n} e^{2\alpha\tau_j^*}\hat{\phi}_j^2\left(\hat{e}_j\left(t-\tau_j(t)\right)\right). \qquad (15)
\end{aligned}
$$

By applying (12)–(15) to (11), we obtain

$$
\begin{aligned}
\dot{V}(t) &\le \hat{e}(t)^T\left(-\widetilde{C}^T P - P\widetilde{C} + P(AA^T + BB^T)P\right.\\
&\qquad\left. +(K_1+K_2)\right)\hat{e}(t)\\
&= \hat{e}(t)^T\left(-\widetilde{C}^T P - P\widetilde{C} + PDD^T P + (K_1+K_2)\right)\hat{e}(t)\\
&= -\,\varepsilon\hat{e}(t)^T\hat{e}(t). \qquad (16)
\end{aligned}
$$

According to Lyapunov theory, the last inequality $\dot{V}(t) \le -\varepsilon\|\hat{e}(t)\|^2$ indicates $V(t)$ converges to zero asymptotically as well as $\hat{e}(t) = 0$ is asymptotically convergent. By Lemma 2, we conclude $e(t)$ converges to zero globally and exponentially with a rate of α, i.e. $\|e(t)\| \le \beta(\alpha)\exp^{-\alpha t}(\sup_{-\tau\le s\le 0}\|\psi(s)-\varphi(s)\|)$ is satisfied. This completes the proof.

Remark 3: In Lemma 1, it is not apparent how one can choose the matrix Ω such that H has no eigenvalues on the imaginary axis. Therefore, it is not simple to find the analytical solutions (if they exit) for the condition of the Main Theorem. Fortunately, they can be solved numerically in almost all cases by an eigenvalue-solver MATLAB and a trial-and-error procedure. Furthermore, the sufficient condition in the Main Theorem would be easily satisfied if $\underset{\max}{\mathrm{Re}}\{\lambda_i(-\widetilde{C})\}$ (the maximum real part of all eigenvalues of $-\widetilde{C}$) is more negative by suitably selecting the gain matrix Ω.

Remark 4: To obtain the gain matrix Ω in the proposed controller (4), a computational procedure is proposed as follows.

- Step 1: Given a positive constant α and an arbitrarily sufficiently small positive constant ε, choose a suitable gain matrix Ω by using any eigenvalue assignment technique such that $-\widetilde{C}$ is a stable matrix.
- Step 2: Construct the Hamiltonian matrix H in Lemma 1 and check if H has no eigenvalues on the imaginary axis. If so, then the procedure goes to Step 4. Otherwise, the procedure continues to Step 3.
- Step 3: Tune the value of $\underset{\max}{\mathrm{Re}}\{\lambda_i(-\widetilde{C})\}$ more negative by selecting a new gain matrix Ω and go back to Step 2.
- Step 4: Obtain the state-feedback controller (4).

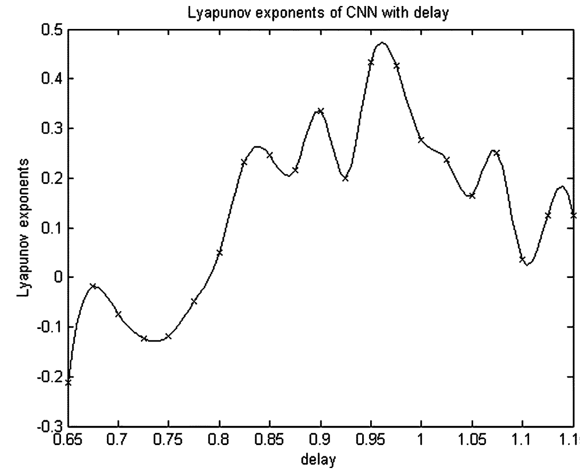

Fig. 1. Largest Lyapunov exponent of Example 1 versus the delay parameter.

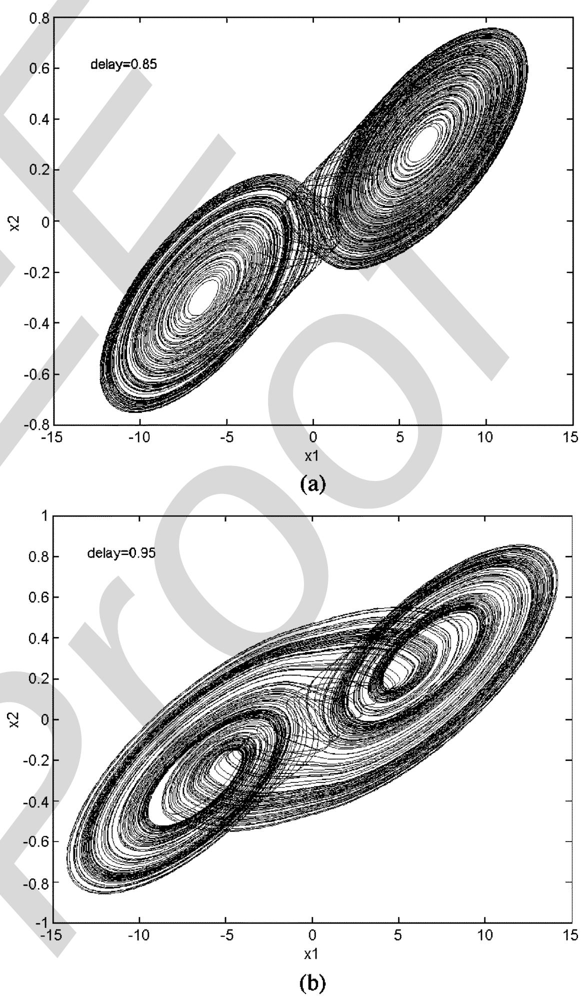

Fig. 2. $x_1 - x_2$ plot for (a) $\mathrm{delay} = 0.85$ and (b) $\mathrm{delay} = 0.95$ in Example 1.

IV. Illustrative Examples

To demonstrate the validity of the exponential synchronization condition, three examples are given in this section. The first one involves synchronizing the cellular neural networks with time-varying delays; the others involve synchronizing the Hopfield neural networks with time-varying delays.

Example 1: A two-dimensional cellular neural network with time-varying delays is given in [5] and described by the following equation:

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{2} a_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{2} b_{ij} f_j\left(x_j\left(t-\tau_j(t)\right)\right), \quad i=1,2 \quad (17)$$

where $c_i = 1$, $A = (a_{ij})_{2\times 2} = \begin{bmatrix} 1+\frac{\pi}{4} & 20 \\ 0.1 & 1+\frac{\pi}{4} \end{bmatrix}$, $B = (b_{ij})_{2\times 2} = \begin{bmatrix} -\sqrt{2}\frac{\pi}{4}1.3 & 0.1 \\ 0.1 & -\sqrt{2}\frac{\pi}{4}1.3 \end{bmatrix}$ and $f_i(x_i) = 0.5(|x_i+1| - |x_i-1|)$, respectively. The delays $\tau_1(t) = \tau_2(t) = (1-e^{-t})/(1+e^{-t})$ are time-varying and satisfy $0 \le \tau_j(t) \le 1 = \tau_j^*$, $0 \le \dot{\tau}_j(t) \le (1/2)j^*$, $j = 1, 2$. The chaotic behavior of the system with delay varying form 0.845 to 1 has been reported in [5]. We employ the algorithm proposed by Wolf *et al.* [24] to determine Lyapunov exponents for the system (17). Fig. 1 shows the largest Lyapunov exponents of (17) by varying the delay parameter from 0.65 to 1.15. Fig. 2(a) and (b) shows the $x_1 - x_2$ plot with the initial condition $[0.1 \;\; 0.1]^T$ for delay 0.85 and 0.95, respectively. To achieve synchronization, the response system is designed as follows:

$$\dot{z}_i(t) = -c_i z_i(t) + \sum_{j=1}^{2} a_{ij} f_j\left(z_j(t)\right) + \sum_{j=1}^{2} b_{ij} f_j\left(z_j\left(t-\tau_j(t)\right)\right) + u_i(t), \quad i=1,2. \quad (18)$$

The system satisfies assumption (H) with $L_1 = L_2 = 1$. According to the ***Main Theorem***, we can design $\omega_1 - \omega_2$ parameter space of the controller gain matrix $\Omega = (\omega_{ij})_{2\times 2} = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix}$ by the computational procedure given in Remark 4 so that the matrix $-\widetilde{C} = \begin{bmatrix} -1+\alpha-\omega_1 & 0 \\ 0 & -1+\alpha-\omega_2 \end{bmatrix}$ is stable and the Hamiltonian matrix H with $K_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $K_2 = (e^{2\alpha\tau_j^*}/(1-r_j^*))\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $DD^T = \begin{bmatrix} 405.2826 & 35.5977 \\ 35.5977 & 5.2926 \end{bmatrix}$ has no eigenvalues on the imaginary axis at least for $\alpha = 0.1$ and $\varepsilon = 10^{-5}$. The region of the parameters ω_1 and ω_2 is depicted in Fig. 3. If the controller gain matrix is chosen as $\Omega = \begin{bmatrix} 40 & 0 \\ 0 & 60 \end{bmatrix}$, it is easily found that the matrix $-\widetilde{C}$ is stable and the Hamiltonian matrix H has no eigenvalues on the imaginary axis for the given $\alpha = 0.2174$. Fig. 4 depicts the synchronization error with the initial conditions $x(s) = [0.1 \;\; 0.1]^T$ and $z(s) = [0.2 \;\; -0.2]^T$ for $(-(1-e^{-t})/(1+e^{-t})) \le s \le 0$, respectively.

Example 2: A Hopfield neural networks with time-varying delays is given in [7] and its dynamics is expressed by

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{2} a_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{2} b_{ij} f_j\left(x_j\left(t-\tau_j(t)\right)\right), \quad i=1,2 \quad (19)$$

where $c_i = 1$, $A = (a_{ij})_{2\times 2} = \begin{bmatrix} 2 & -0.1 \\ -5 & 2 \end{bmatrix}$, $B = (b_{ij})_{2\times 2} = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -1.5 \end{bmatrix}$ and $f_i(x_i) = \tanh(x_i)$, respectively. The delays $\tau_1(t) = \tau_2(t) = 0.6(1-\cos(t))$ are time-varying and satisfy $0 \le \tau_j(t) \le 1.2 = \tau_j^*$, $-0.6 \le \dot{\tau}_j(t) \le 0.6 = r_j^*$, $j = 1, 2$. The chaotic behavior of the system with constant delay $\tau_1 = \tau_2 = 1$ has been pointed out [7]. Fig. 5 shows the largest Lyapunov exponents of (19) by varying the delay parameter from 0 to 1.2. Fig. 6(a) and (b) shows the $x_1 - x_2$ plot with the initial condition $[0.4 \;\; 0.6]^T$ for delay 1 and

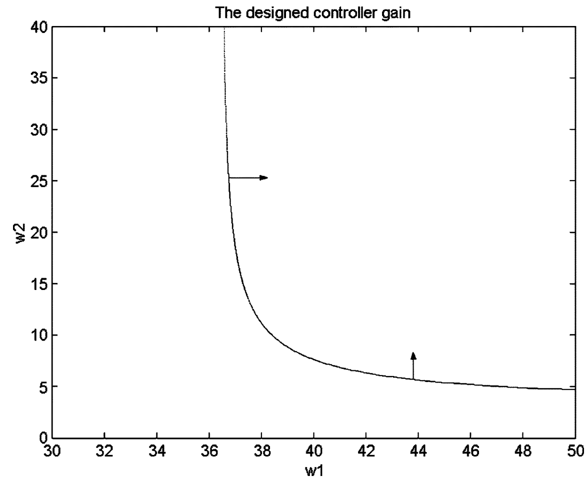

Fig. 3. Designed $\omega_1 - \omega_2$ parameter space for a given $\alpha = 0.1$ (Example 1).

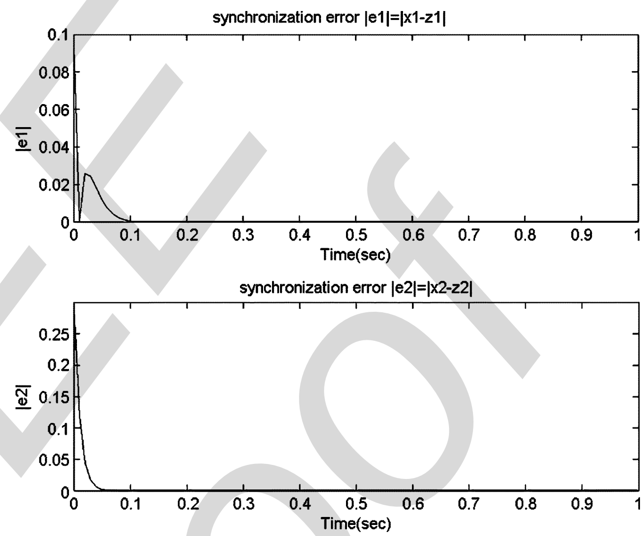

Fig. 4. Synchronization error for $\tau = (1-e^{-t})/(1+e^{-t})$ (Example 1).

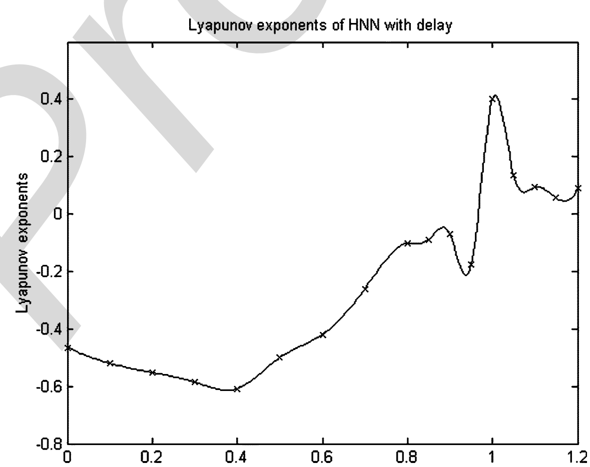

Fig. 5. Largest Lyapunov exponent of Example 2 versus the delay parameter.

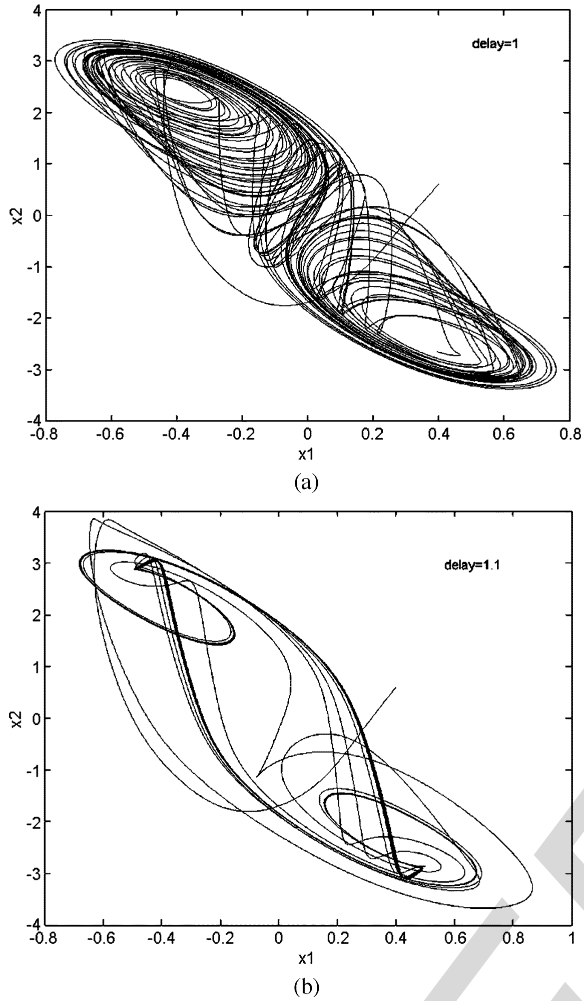

Fig. 6. $x_1 - x_2$ plot for (a) $\text{delay} = 1$ and (b) $\text{delay} = 1.1$ in Example 2.

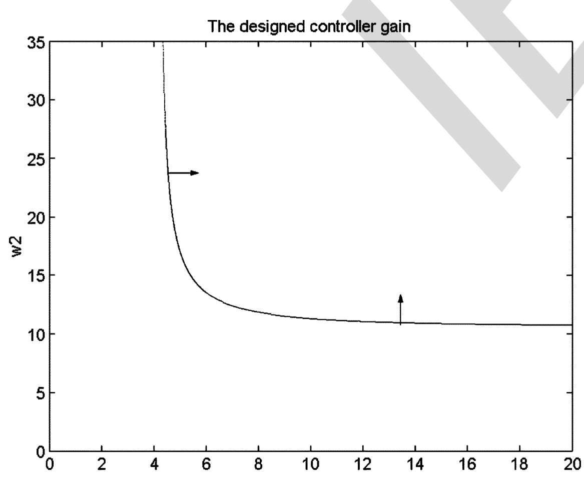

Fig. 7. Designed $\omega_1 - \omega_2$ parameter space for a given $\alpha = 0.1$ (Example 2).

1.1, respectively. To achieve synchronization, the response system is designed as follows:

$$\dot{z}_i(t) = -c_i z_i(t) + \sum_{j=1}^{2} a_{ij} f_j\left(z_j(t)\right)$$

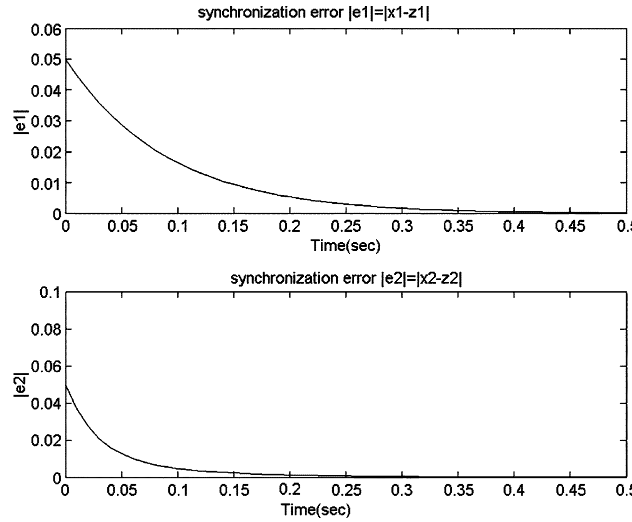

Fig. 8. Synchronization error for $\tau = 0.6(1 - \cos(t))$ (Example 2).

$$+ \sum_{j=1}^{2} b_{ij} f_j\left(z_j\left(t - \tau_j(t)\right)\right) + u_i(t), \quad i = 1, 2. \quad (20)$$

The system satisfies assumption (H) with $L_1 = L_2 = 1$. According to the ***Main Theorem***, we apply the computational procedure given in Remark 4 to design $\omega_1 - \omega_2$ parameter space of the controller gain matrix $\Omega = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix}$ so that. the matrix $-\widetilde{C} = \begin{bmatrix} -1+\alpha-\omega_1 & 0 \\ 0 & -1+\alpha-\omega_2 \end{bmatrix}$ is stable and the Hamiltonian matrix H with $K_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $K_2 = (e^{2\alpha\tau_j^*}/(1 - r_j^*)) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $DD^T = \begin{bmatrix} 6.27 & -9.75 \\ -9.75 & 31.29 \end{bmatrix}$ has no eigenvalues on the imaginary axis at least for the given $\alpha = 0.1$ and $\varepsilon = 10^{-5}$. The region of the parameters ω_1 and ω_2 is depicted in Fig. 7. If the controller gain matrix is chosen as $\Omega = \begin{bmatrix} 10 & 0 \\ 0 & 30 \end{bmatrix}$, it is easily found that the matrix $-\widetilde{C}$ is stable and the Hamiltonian matrix H has no eigenvalues on the imaginary axis for the given $\alpha = 0.6387$ and $\varepsilon = 10^{-5}$. Fig. 8 depicts the synchronization error with the initial conditions $x(s) = [0.4 \;\; 0.6]^T$ and $z(s) = [0.35 \;\; 0.65]^T$ for $-0.6(1 - \cos(t)) \leq s \leq 0$, respectively.

Example 3: A four-dimensional Hopfield neural networks with time-varying delays is given as follows:

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{4} a_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{4} b_{ij} f_j\left(x_j\left(t - \tau_j(t)\right)\right), \quad i = 1, 2, 3, 4 \quad (21)$$

where $c_i = 1$, $A = (a_{ij})_{4\times 4} = \begin{bmatrix} 0.85 & -2 & -0.5 & 0.5 \\ 1.8 & 1.15 & 0.6 & 0.3 \\ 1.1 & 1.21 & 2.5 & 0.05 \\ 0.1 & -0.4 & -1.5 & 1.45 \end{bmatrix}$ and $f_i(x_i) = \tanh(x_i)$, respectively. The chaotic behavior of the system with $B = (b_{ij})_{4\times 4} = (0)_{4\times 4}$ and $\tau_j(t) = 0$ has been pointed out in [25]. In this example, we choose $B = (b_{ij})_{4\times 4} = \begin{bmatrix} -0.5 & -1 & 0.5 & 0.2 \\ 0.4 & -0.1 & 0.3 & 0.3 \\ 0.1 & -0.8 & 0.2 & 0.05 \\ 0 & -0.4 & -0.6 & -0.5 \end{bmatrix}$ and the time-varying delays as $\tau_j(t) = 0.5(1 - \cos(t))$, $j = 1, 2, 3, 4$, which satisfy

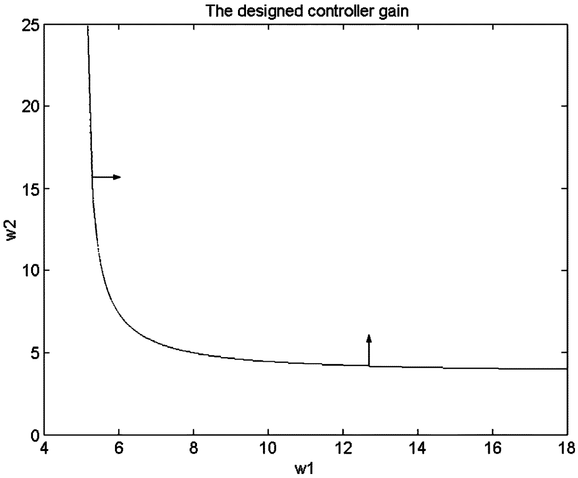

Fig. 9. Designed $\omega_1 - \omega_2$ parameter space for a given $\alpha = 0.1$ (Example 3).

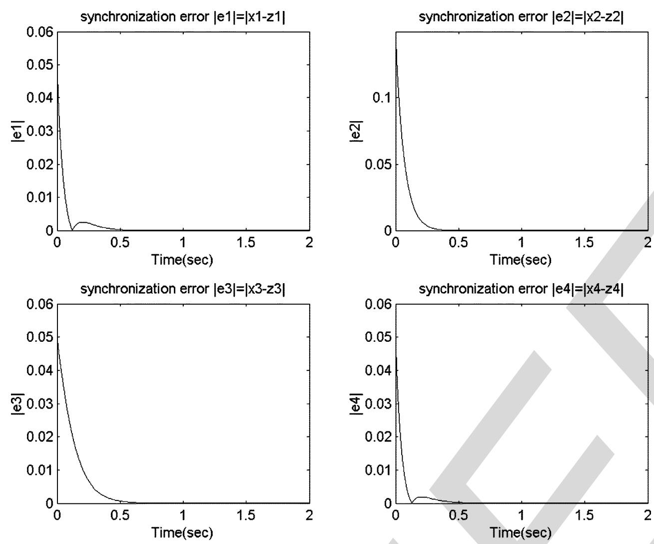

Fig. 10. Synchronization error for $\tau = 0.5(1 - \cos(t))$ (Example 3).

$0 \leq \tau_j(t) \leq 1 = \tau_j^*$, $-0.5 \leq \dot{\tau}_j(t) \leq 0.5 = r_j^*$, $j = 1, 2, 3, 4$. To achieve synchronization, the response system is designed as follows:

$$\dot{z}_i(t) = -c_i z_i(t) + \sum_{j=1}^{4} a_{ij} f_j\left(z_j(t)\right) + \sum_{j=1}^{4} b_{ij} f_j\left(z_j\left(t - \tau_j(t)\right)\right) + u_i(t), \quad i = 1, 2, 3, 4. \quad (22)$$

The system satisfies assumption (H) with $L_1 = L_2 = L_3 = L_4 = 1$. If we choose the synchronization degree $\alpha = 0.1$ and the controller gain matrix $\Omega = diag(\omega_i)$, $i = 1, 2, 3, 4$ with $\omega_1 = \omega_3$ and $\omega_2 = \omega_4$, then the four sub-matrices of H are obtained as follows:

$$K_1 = I_4, \quad K_2 = 2e^{0.2} I_4$$
$$-\widetilde{C} = diag(-1 + 0.1 - \omega_1, -1 + 0.1 - \omega_2, -1 + 0.1 - \omega_1, -1 + 0.1 - \omega_2)$$

and $$DD^{\mathrm{T}} = \begin{bmatrix} 6.7625 & -0.81 & -1.85 & 2.36 \\ -0.81 & 5.3625 & 5.0815 & -1.035 \\ -1.85 & 5.0815 & 9.6191 & -3.8765 \\ 2.36 & -1.035 & -3.8765 & 5.2925 \end{bmatrix}.$$

The controller parameters $\omega_1 - \omega_2$ are determined by the computational procedure so that the matrix $-\widetilde{C}$ is stable and the Hamiltonian matrix H has no eigenvalues on the imaginary axis for the sufficiently small constant $\varepsilon = 10^{-5}$. The region of the parameters ω_1 and ω_2 is depicted in Fig. 9. If the controller parameters are chosen as $\omega_1 = \omega_3 = 10$ and $\omega_2 = \omega_4 = 15$, the synchronization error of the systems (21) and (22) with the initial conditions $x(s) = [0.1 \ \ -0.1 \ \ 0.1 \ \ 0.2]^T$ and $z(s) = [0.15 \ \ 0.05 \ \ 0.15 \ \ 0.25]^T$ for $-0.5(1 - \cos(t)) \leq s \leq 0$, respectively, is shown in Fig. 10.

V. Conclusion

This paper has presented a sufficient condition to guarantee the globally exponential synchronization for a class of neural networks including Hopfield neural networks and cellular neural networks with time-varying delays. A feedback control law is derived to achieve the exponential synchronization of drive-response structure of the chaotic neural networks; and its feedback gain matrix is designed to satisfy a certain Hamiltonian matrix without eigenvalues on the imaginary axis instead of directly solving an algebraic Riccati equation.

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