Generalized projective synchronization of chaotic systems with unknown dead-zone input: Observer-based approach

Yung-Ching Hung, Chi-Chuan Hwang, and Teh-Lu Liaoa
Department of Engineering Science, National Cheng Kung University, Tainan, 701 Taiwan

Jun-Juh Yan
Department of Computer and Communication, Shu-Kung University, Kaohsiung 824, Taiwan

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In this paper we investigate the synchronization problem of drive-response chaotic systems with a scalar coupling signal. By using the scalar transmitted signal from the drive chaotic system, an observer-based response chaotic system with dead-zone nonlinear input is designed. An output feedback control technique is derived to achieve generalized projective synchronization between the drive system and the response system. Furthermore, an adaptive control law is established that guarantees generalized projective synchronization without the knowledge of system nonlinearity, and/or system parameters as well as that of parameters in dead-zone input nonlinearity. Two illustrative examples are given to demonstrate the effectiveness of the proposed synchronization scheme. © 2006 American Institute of Physics. [DOI: 10.1063/1.2336728]

Generally, chaotic systems are nonlinear deterministic systems that possess complex and unpredictable behavior. Chaos control and chaos synchronization as well as its applications have been developed and thoroughly studied over the past two decades. The typical configuration of the synchronization of chaotic systems consists of drive and response systems. The drive system drives the response system via coupling signal(s) to synchronize each other’s behavior. There are several different types of synchronizations have been proposed in the literature such as complete synchronization (CS), phase synchronization (PS), lag synchronization (LS), generalized synchronization (GS), and generalized projective synchronization (GPS). In the past decade, based on the state observer design technique, many observer-based synchronization schemes have been proposed. However, in practice, the values of parameters and nonlinearities that exist in chaotic systems and in the control input are partially known a priori. These parametric uncertainties and nonlinearities will destroy the synchronization performance. In this paper we demonstrate the projective synchronization of chaotic systems without the knowledge of these parametric uncertainties and nonlinearities via a state observer technique and adaptive control scheme.

I. INTRODUCTION

The synchronization of chaotic systems has received great attention over past few years. Several theoretical studies and/or laboratory experiments have demonstrated the pivotal role of this phenomenon in secure communications.1–9 The preliminary configuration of chaos synchronization consists of two chaotic systems: a drive system and a designed response system. The drive system drives the response system via the transmitted signal(s) so that the trajectory of drive system synchronizes that of the response system. Several different types of synchronizations have been proposed in the literature, for example, complete synchronization (CS),10–13 phase synchronization (PS),14,15 lag synchronization (LS),16,17 generalized synchronization (GS),18–23 as well as generalized projective synchronization (GPS).24–36 Complete synchronization, which implies states of the drive and response systems, are synchronized. The phase synchronization occurs in a manner that phases are locked together, but the amplitude may evolve incoherently. Lag synchronization that appears as a coincidence of state of the drive system at lagged time and that of response system at current time. The generalized synchronization means that states of the response system synchronize that of the drive system through a nonlinear smooth functional mapping. Generalized synchronization has been demonstrated by nonidentical chaotic systems and in distinct classes of chaotic systems. Generalized projective synchronization, which belongs to a subclass of the generalized synchronization, unifies different types of synchronization phenomena such as antiphase synchronization and complete synchronization, as well as a specific case of generalized synchronization into one by a scaling factor. The performance of projective synchronization can be selected and manipulated by controlling the scaling factor.24–36

However, from the viewpoint of control theory, the response system can be considered an observer of the drive system. All the state variables of the response system are constructed by a transmitted signal from the drive system, and then, all or partial state trajectories of these systems are driven into synchronization through an appropriate control design for the observer. Based on the observer design technique, many observer-based synchronization schemes have been proposed.30,32,37 The control schemes with state feedback or output feedback for chaos synchronization can be
realized by electronic components such as operational amplifiers (OPA), resistor, and capacitor, etc. or by electromechanical actuators. However, in practice, there always exist nonlinearities, including the saturation, backlash, and dead zone in OPA-based circuits or electromechanical devices. Furthermore, it is difficult to maintain the exact values of resistance and capacitance due to the uncontrollable environmental conditions and to know the exact parameters (e.g., width and slope) of the dead zone. These input nonlinearities and parametric uncertainties may lead to serious degradation of system performance and cause system instability. Therefore, it is clear that the effects of input nonlinearities with unknown parameters must be taken into account when analyzing and implementing the synchronization control scheme.

Motivated by the aforementioned results, an observer-based response system with a dead zone in control input is designed to ensure the synchronization between the drive and response systems coupled with a scalar transmitted signal. Both nonadaptive and adaptive output feedback control techniques are employed to obtain the generalized projective synchronization of drive-response chaotic systems. Two well-known Chua’s system and Van der Pol-Duffing oscillator are given as illustrative examples to demonstrate the effectiveness of the proposed projective synchronization scheme.

**Notation:** Note that throughout the remainder of this paper, the notation \( M^T \) denotes the transpose of a square matrix \( M \), while for \( x \in \mathbb{R}^n \), \( \|x\| = (x^T x)^{1/2} \) denotes the Euclidean norm of the vector; \( P > 0 \) denotes \( P \) is a symmetric positive definite symmetric matrix and \( P = P^T < 0 \) denotes \( P \) is a symmetric negative definite symmetric matrix. \( \lambda_{\text{min}}(P) \) denotes the smallest eigenvalue of the matrix \( P \).

**II. PROBLEM FORMULATION**

Generally, dynamics of many chaotic systems can be decomposed into below two parts: a linear dynamics with respect to state and a nonlinear feedback part with respect to the system output. Therefore, a class of chaotic systems is considered here and its dynamics are described by the following form:

\[
\dot{x} = Ax + f(y) + B[\theta^T g(x, y)], \quad y = C^T x, \tag{1}
\]

where \( \cdot^T \) denotes the vector transpose; \( y \in \mathbb{R} \) denotes the system output; \( x \in \mathbb{R}^n \) represents the state vector, and \( A, B, \) and \( C \) denote known system matrices with appropriate dimensions. The pair \((A, B)\) is controllable, i.e., the controllability matrix \([B, AB, \ldots, A^{n-1}B]\) is in full rank, and, furthermore, the pair \((C^T, A)\) is observable, namely, the observability matrix \([C, A^T C, \ldots, (A^T)^{n-1} C]\)^T is in full rank. Assuming that \( \theta \in \mathbb{R}^p \) and \( g \in \mathbb{R}^q \) are real analytic vectors with \( f(0) = 0 \) and \( g(0, 0) = 0 \), respectively. Furthermore, we assume that the system (1) has a unique solution \( x(t) \) passing through the initial state \( x(0) = x_0 \) and this solution is well defined over an interval \([0, \infty)\).

For the class of systems (1), we make the following assumption.

**III. OBSERVER-BASED CHAOS SYNCHRONIZATION SCHEMES**

On the basis of state observer design, a response system in the driver-response configuration of chaos synchronization corresponding to (1) is given as follows:

\[
\dot{x} = Ax + \lambda f(y) + L(\lambda y - \hat{y}) - B \phi(u), \quad \hat{y} = C^T \hat{x}, \tag{4}
\]

where \( \hat{x} \) denotes the state of the response system and the constants vector \( L \in \mathbb{R}^p \) is chosen such that \((A - LC)^T\) is an exponentially stable matrix. Here \( u \) is the control input that will be appropriately designed to obtain the generalized projective synchronization between (1) and (4) subject to the system’s nonlinearities and/or unknown parameters.

With the coupled systems (1) and (4), the generalized projective synchronization can be characterized by synchronizing \( \lambda x \) and \( \hat{x} \) asymptotically, i.e., \( \lim_{t \to \infty} \| \lambda x - \hat{x} \| = 0 \), where \( \lambda \) is called the “scaling factor.” Moreover, to imple-
A. Nonadaptive control scheme

If all constants $\theta_i, i=1,\ldots,p$, and $\sigma^*$ in (2) and (7) are well known, the control input for the generalized projective synchronization is derived as follows:

$$u = - \begin{bmatrix} \lambda \left( \sum_{i=1}^{p} \theta_i |\tilde{g}(y)| \right) + \frac{\sigma^*}{m} \end{bmatrix} \text{sign}(e_t). \quad (9)$$

Consider the stability of the error dynamical system (8) under the control law (9); we choose a Lyapunov function as

$$V(t) = \frac{1}{m} e^T(t) P e(t), \quad (10)$$

where $P=P^T > 0$ satisfies Eq. (3). It is obvious that $V$ is a positive and decrescent function. Furthermore, $V$ is radically unbounded.

Taking the time derivative of $V$ along the trajectories of the resulting error dynamical system (8) with (9) and applying the K-Y lemma (3) and property of nonlinear function $h[u(t)]$ (7) yields

$$\dot{V} = \frac{1}{m} \left[ e^T(A-LC^T)^T P + P(A-LC^T) e \right] + \frac{2}{m} (e^T PB [\lambda \tilde{g}(x,y)] + m u + h(u))$$

$$\quad \leq -\frac{1}{m} (e^T Q e) + \frac{2}{m} \left( \lambda \|e_t\| \left( \sum_{i=1}^{p} \theta_i |\tilde{g}(y)| \right) + 2e_t u + \frac{2}{m} |e_t| \sigma^* \right)$$

$$\quad = -\frac{1}{m} (e^T Q e) \leq 0. \quad (11)$$

From (11), we have $\dot{V}$, which is negative semidefinite, and it follows that the equilibrium points $e=0$ of system (8) are uniformly globally stable, i.e., $e(t) \in L_{\infty}$ [$e(t)$ is a bounded signal for all $t \geq 0$]. Moreover, from (8) and (11), we can easily show that $\tilde{e}(t) \in L_{\infty}$ and $e(t) \in L_2$ [the square of $e(t)$ is integrable with respect to time]. Therefore, by Barbala’s lemma, we conclude that $e(t) \to 0$ as $t \to \infty$, which means that for the drive chaotic system (1) satisfying the (A1) and (A2) and the observer-based response system (4) with the dead zone in input, the proposed control law (9) will achieve the generalized projective synchronization asymptotically, i.e., $\|e(t)\| = \|\lambda x(t) - \tilde{x}(t)\| \to 0$ as $t \to \infty$ for all initial conditions. Furthermore, all signals inside the closed-loop system remain bounded.

B. Adaptive control scheme

The control law derived thus far requires that the knowledge of the system’s parameters $\theta_i^*, i=1,\ldots,p$, and parameter $\sigma^*$ of the dead-zone model. However, in many real applications it is difficult to exactly determine both the values of the system’s parameters $\theta_i^*, i=1,\ldots,p$, and parameter $\sigma^*$ of the dead-zone nonlinearity. Consequently, the control law $u$ given in (9) cannot be easily obtained to ensure the generalized projective synchronization. To overcome these draw-
backs, let us redefine the control parameters in (9) as follows: \( \rho_i = \Theta_i / m \) and \( \eta = \sigma / m \), and an adaptive control law with \( \hat{\Theta}_i, i=1, \ldots, p \), and \( \hat{\sigma} \), where \( \hat{\Theta} \) and \( \hat{\sigma} \) are estimates of \( \rho_i \) and \( \eta \), is designed to derive the adaptive response system, and thereby achieving the generalized projective synchronization between the drive system and the response system. Now, similar to (9), the adaptive control law with the estimates of \( \hat{\Theta}_i, i=1, \ldots, p \), and \( \hat{\sigma} \) is derived as follows:

\[
\hat{\Theta}_i = \left| \lambda \right| |e_i| \hat{|g_i(y)|}, \quad \text{with} \quad \hat{\Theta}_i(0) \geq 0, \quad i = 1, \ldots, p, \tag{13}
\]

\[
\hat{\sigma} = \gamma_i |e_i|, \quad \text{with} \quad \hat{\sigma}(0) \geq 0, \tag{14}
\]

where \( \gamma_i \) is a positive constant specified by the designer. The proposed adaptive control scheme will guarantee the globally asymptotic stability of the error system (8) and achieve the synchronization objective.

Let us first define the following parameter errors: \( \hat{\Theta}_i = \rho_i - \hat{\Theta}_i, i=1, \ldots, p \), and \( \sigma = \eta - \hat{\sigma} \); then consider a Lyapunov function as follows:

\[
V(t) = \frac{1}{m} e^T(t)P e(t) + \sum_{i=1}^{p} \hat{\Theta}_i^2(t) + \frac{1}{\gamma_i} \sigma^2(t). \tag{15}
\]

It is obvious that \( V \) is a positive and decrescent function. Furthermore, \( V \) is radially unbounded.

Taking the time derivative of \( V \) along the trajectories of the resulting error dynamical system (8) with (9) and applying (3) and (7) leads to

\[
\dot{V} = \frac{1}{m} e^T((A-LC^T)^TP + P(A-LC^T))e + \frac{2}{m} e^TPB[\lambda(\theta^T g(x,y))]e + \sum_{i=1}^{p} \hat{\Theta}_i \dot{\hat{\Theta}}_i + \frac{2}{\gamma_i} \sigma \dot{\sigma}.
\]

Since \( \rho_1 \) is constant, \( \dot{\rho}_1 = 0 \) and the following expression holds:

\[
\dot{\hat{\Theta}}_i = -\hat{\Theta}_i, \quad i = 1, \ldots, p, \tag{17}
\]

and \( \sigma \) is constant, \( \dot{\eta} = 0 \), and the following expression holds:

\[
\dot{\hat{\sigma}} = \hat{\sigma}. \tag{18}
\]

Substituting (13), (14), (17), and (18) into (16), we can get the following result:

\[
\dot{V} = \frac{1}{m} (e^T Q e) + \frac{2}{m} e_1^T \lambda(\theta^T g(x,y)) + mu + h(u) + \frac{2}{m} \sum_{i=1}^{p} \hat{\Theta}_i \dot{\hat{\Theta}}_i + \frac{2}{\gamma_i} \sigma \dot{\sigma}.
\]

From (19), we have \( \dot{V} \), which is negative semidefinite, and it follows that the overall systems are uniformly stable, i.e., \( e(t) \in L_\infty \), \( \theta(t) \in L_\infty \), and \( \sigma(t) \in L_\infty \), as well as \( u(t) \in L_2 \). Moreover, from (8) and (19), we can easily show that \( e(t) \in L_\infty \) and \( e(t) \in L_2 \). Therefore, from Barbalat’s lemma, we conclude that \( e(t) \to 0 \) as \( t \to \infty \), which means that for the drive chaotic system (1) satisfying the (A1) and (A2) and the observer-based response system (4) with the dead zone in control input, the proposed adaptive control law (12) with adaptation laws (13) and (14) will achieve the generalized projective synchronization asymptotically, i.e., \( \| e(t) \| = \| k x(t) - \hat{x}(t) \| \to 0 \) as \( t \to \infty \) for all initial conditions. Furthermore, all signals inside the closed-loop system remain bounded.

## IV. ILLUSTRATIVE EXAMPLES

To illustrate the effectiveness of the proposed adaptive synchronization scheme, two examples of well-known chaotic systems: Chua’s system and Van der Pol-Duffing oscillator are considered and their numerical simulations are performed.
Example 1 Chua’s system:
A nondimensional differential equation for Chua’s system is given by
\[
\begin{align*}
\dot{x}_1 &= \alpha [x_2 - x_1 - f(x_1)], \\
\dot{x}_2 &= x_1 - x_2 + x_3, \\
\dot{x}_3 &= -\beta_1 x_2 - \beta_2 x_3,
\end{align*}
\] (20)
and
\[
f(x_1) = bx_1 + 0.5 (a - b) (|x_1 + 1| - |x_1 + 1|),
\] (21)
where \(\alpha, \beta_1, \) and \(\beta_2\) are system parameters and are given as follows: \(\alpha = 10, \beta_1 = 15,\) and \(\beta_2 = 0.0385; f(x_1)\) denotes a three-segment piecewise linear function in which \(a, b\) are two negative real constants and \(-1 < b < 0.\) Herein, the parameters \(a\) and \(b\) are assumed to be unknown. By introducing \(y = x_1,\) Eq. (20) can be rewritten in a compact form as follows:
\[
\begin{align*}
\dot{X} &= \begin{bmatrix} -10 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & -15 & -0.0385 \end{bmatrix} X + \begin{bmatrix} 1 \\ -10b y - 5(a-b)(|y+1| - |y+1|) \end{bmatrix} \\
&= A X + B [\theta^T g(y)],
\end{align*}
\] (22)
\[
y = x_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} C^T X,
\] (23)
where \(\theta^T = [\theta_1 \theta_2] = [-10b -5(a-b)]\) and \(g(y) = [g_1(y) \ g_2(y)] = [y \ |y+1| - |y+1|].\) It can be easily verified that \((A, B)\) is a controllable pair and \((C^T, A)\) is an observable pair. Also, the vector \(L^T = [-7 \ 1.9536 \ 0.7343]\) can be

FIG. 3. (a) Trajectories of state error between the drive and response system when \(\lambda = 1.\) (b) Trajectories of state error between the drive and response system when \(\lambda = -1.\)
found so that the eigenvalues of matrix \((A-LC_t)\) are 
\(-2.3532\) and \(-1.0159 \pm 4.8231\), and that the transfer function 
\(H(s) = C\left[I - (A-LC_t)\right]^{-1}B = (s^3 + 1.385s + 15.38)/(s^3 + 4.385s^2 + 29.085s + 57.17)\) is strictly positive real. Hence, 
(A1) and (A2) are satisfied. Moreover, the symmetric and 
positive definite matrices \(P\) and \(Q\) satisfying Eq. (3) are
\[
P = \begin{bmatrix}
1 & 0 & 0 \\
0 & 11 & -0.6667 \\
0 & -0.6667 & 0.8658
\end{bmatrix},
\]
\(Q = \begin{bmatrix}
6 & 0.0001 & 0 \\
0.0001 & 2 & 1.0637 \\
0 & 1.0637 & 2
\end{bmatrix}.
\]
(24)
As derived earlier, an observer-based response system is de-
signed as follows:
\[
\dot{x} = \begin{bmatrix}
-10 & 10 & 0 \\
1 & -1 & 1 \\
0 & -15 & -0.0385
\end{bmatrix}x + L(\lambda y - \hat{y}) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \phi(u),
\]
(25)
and the adaptive control law is designed as follows:
\[
u = -\left(\lambda|\hat{\theta}_1|y + \hat{\theta}_2(|y|+1) - |y+1|\right) + \hat{\sigma} \text{sign}(e),
\]
(26)
where the estimate \(\hat{\theta}_1\) and \(\hat{\theta}_2\) are updated according to fol-
lowing algorithm:
\[
\hat{\theta}_1 = |\lambda||e_1||y|,
\]
(27)
\[
\hat{\theta}_2 = |\lambda||e_1||y+1| - |y+1|, \quad \hat{\sigma} = |e_1|,
\]
and the nonlinear input containing a dead zone is defined as

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig4.png}
\caption{(a) Trajectories of estimated parameters when \(\lambda = 1\). (b) Trajectories of estimated parameters when \(\lambda = -1\).}
\end{figure}
\[ \phi[u(t)] = \begin{cases} 
(u(t) - u_+)(1 - 0.3 \sin[u(t)]), & u(t) > u_+ \\
0, & u_- \leq u(t) \leq u_+,
(u(t) - u_-)(1 - 0.3 \cos[u(t)]), & u(t) < -u_-.
\end{cases} \quad (28) \]

As given in (5) and (6), we can obtain \( m = 0.7, \sigma_1[u(t)] = 0.3u_\sin[u(t)] \) and \( \sigma_2[u(t)] = 0.3u_\cos[u(t)] \). In numerical simulations, the system’s parameters are chosen as \( a = -1.28, b = -0.69 \), thereby implying \( \beta_1 = 6.9 \) and \( \beta_2 = 2.95 \).

The parameters in the dead zone are chosen as follows: \( u_+ = 5 \) and \( u_- = -3 \). Figures 3(a) and 3(b) show the error trajectories of the generalized projective synchronization for Chua’s circuits with the scaling factors \( \lambda = 1 \) and \( \lambda = -1 \), respectively. The initial values of the update law is given as \( \hat{\theta}_1(0) = 0, \hat{\theta}_2(0) = 0, \) and \( \hat{\sigma}(0) = 0 \). The initial states of the drive and response system are given as \( [x_{d1}(0), x_{d2}(0), \dot{x}_{d3}(0)] = [0.1, 0.1, 0.1] \), \( [x_{r1}(0), x_{r2}(0), \dot{x}_{r3}(0)] = [-3, -2] \), respectively.

Figures 4(a) and 4(b) show trajectories of the estimated parameters when \( \lambda = 1 \) (complete synchronization) and \( \lambda = -1 \) (antiphase synchronization), respectively.

**Example 2 Van der Pol-Duffing oscillator:**

In this example, we consider the chaotic synchronization of two unidirectionally coupled Van-der-Pol-Duffing oscillators.\(^{40-42}\) The drive generators described by system of dimensionless differential equations:

\[
\dot{x}_1 = -v(x_1^3 - \alpha x_1 - x_2), \quad \dot{x}_2 = x_1 - x_2 - x_3, \quad \dot{x}_3 = \beta x_2,
\]

where \( \alpha, \beta, \) and \( v \) are three positive real parameters. The system exhibits bistable chaotic dynamics if the parameters are set equal to \( \alpha = 0.35, \beta = 300, \) and \( v = 100 \).

Herein, the parameters \( \alpha \) and \( v \) are perturbed by \( \Delta \alpha \) and \( \Delta v \), respectively. By introducing \( y = x_1 \), Eq. (29) can be rewritten in a compact form as follows:

![FIG. 5. (a) Trajectories of state error between the drive and response system when \( \lambda = 1 \). (b) Trajectories of state error between the drive and response system when \( \lambda = -1 \).](image-url)
\[
\dot{x} = \begin{bmatrix}
35 & 100 & 0 \\
1 & -1 & -1 \\
0 & 300 & 0
\end{bmatrix} x + \begin{bmatrix}
-100y^3 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} \\
\times [\Delta y^3 + (100\Delta \alpha + 0.35\Delta \nu + \Delta \alpha \Delta \nu) y + \Delta \nu x_2]
\]

\[= Ax + f(y) + B[\theta^T g(x, y)], \quad (30)\]

\[y = x_1 = \begin{bmatrix}
1 & 0 & 0
\end{bmatrix} x = C^T x, \quad (31)\]

where the parameter defined as \(\theta^T = [\theta_1 \ \theta_2 \ \theta_3]\) and \(g(x, y)^T = [g_1(x, y) \ \ g_2(x, y) \ \ g_3(x, y)] = [y^3 \ \ y \ \ x_2]\). It can be easily verified that \((A, B)\) is a controllable pair and \((C^T, A)\) is an observable pair. Also, the vector \(L^T = [36 \ \ 1.3333 \ \ -0.3311]\) can be found so that the eigenvalues of matrix \((A - LC)\) are \(-0.9993\) and \(-0.5003 \pm 18.2505\), and that the transfer function \(H(s) = C^T[sI - (A - LC)]^{-1}B = (s^2 + s + 300)/(s^3 + 4s^2 + 334.3s + 333.1)\) is strictly positive real. Hence, \((A1)\) and \((A2)\) are satisfied. Moreover, the symmetric and positive definite matrices \(P\) and \(Q\) satisfying Eq. (3) are
As derived earlier, an observer-based drive system is designed as follows:

\[
P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 301 & 1 \\ 0 & 1 & 1.0067 \end{bmatrix},
\]

\[
Q = \begin{bmatrix} 2 & -0.0078 & 0 \\ -0.0078 & 2 & -0.01 \end{bmatrix}.
\]

(32)

and the control law is designed as follows:

\[
u = -\left(\left[\lambda [\hat{\theta}_1 |y|^3] + \hat{\theta}_2 |y| + \hat{\theta}_3 + \sigma|\text{sign}(\epsilon_1)\right] \right.\]

(34)

where the estimate \(\hat{\theta}_1, \hat{\theta}_2, \) and \(\hat{\theta}_3\) are updated according to the following algorithm:

\[
\dot{\hat{\theta}_1} = \lambda |e_1| |y|^3, \quad \dot{\hat{\theta}_2} = \lambda |e_1| |y|, \quad \dot{\hat{\theta}_3} = \lambda |e_1|, \quad \dot{\hat{\sigma}} = |e_1|,
\]

(35)

and the nonlinear input containing a dead zone is defined as

\[
\phi[u(t)] = \begin{cases} (u(t) - u_+)(1 - 0.3 \sin[u(t)]) & u(t) > u_+, \\ 0 & u_- \leq u(t) \leq u_+, \\ (u(t) - u_-)(1 - 0.3 \cos[u(t)]) & u(t) < -u_. \end{cases}
\]

(36)

As given in (5) and (6), we can obtain \(m=0.7\), \(\sigma_1[u(t)] = 0.3u_0 \sin[u(t)]\) and \(\sigma_2[u(t)] = 0.3u_0 \cos[u(t)]\). In the numerical simulations, the parametric perturbations \(\Delta \alpha\) and \(\Delta \upsilon\) are chosen as \(\Delta \alpha = -0.01\) and \(\Delta \upsilon = 1\), respectively, thereby implying \(\hat{\theta}_1 = 0.06, \hat{\theta}_2 = -0.06,\) and \(\hat{\theta}_3 = 1\). Figures 5(a) and 5(b) show the error trajectories of the generalized projective synchronization for Van-der-Pol-Duffing oscillators with the scaling factors \(\lambda = 1, \lambda = -1,\) respectively. In numerical simulations, the parameters are specified as follows: (i) The parameters in the dead-zone model is given as \(u_+ = -5\) and \(u_- = 3\). (ii) The initial value of update law is given as \(\hat{\theta}_1(0) = 0, \hat{\theta}_2(0) = 0, \hat{\theta}_3(0) = 0,\) and \(\hat{\sigma}(0) = 0\). (iii) The initial value of the drive and response system is given as \(x_{d1}(0) = x_{d2}(0) = x_{d3}(0) = \left[0.1 -0.1 0.1\right]^T\) and \(x_{r1}(0) = x_{r2}(0) = x_{r3}(0) = \left[2 -2 1\right]^T,\) respectively. Figures 6(a) and 6(b) show the trajectories of estimated parameters when \(\lambda = 1\) (complete synchronization) and \(\lambda = -1\) (antiphase synchronization), respectively.

V. CONCLUSIONS

In this work, both nonadaptive and adaptive observer-based approaches have been developed to resolve the generalized projective synchronization problem for a class of chaotic systems in the presence of system’s parametric uncertainties and dead-zone nonlinear in control input. Both drive and response systems are coupled through a scalar transmitted signal. Given certain structural conditions of the drive chaotic system, an observer-based response system and two control schemes are constructed so that those drive system and response system are to be synchronized. Furthermore, the knowledge of parameters in the dead zone and that of the system’s parametric uncertainties are not necessary in the adaptive control design. The synchronization and stability of the overall systems are guaranteed by the Lyapunov stability theory. Numerical simulations of two well-known chaotic systems: Chua’s circuit and Van der Pol-Duffing oscillator are given to demonstrate the effectiveness of the proposed approach.