Robust chaos synchronization of noise-perturbed chaotic systems with multiple time-delays

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Abstract

The aim of this paper is to propose an output coupling and feedback scheme, which is not only to guarantee the asymptotic synchronization between the master and the slave chaotic systems with multiple time-delays but also to attenuate the effects of noise perturbation on the overall error system to a prescribed level in terms of the performance index $H_{\infty}$-norm. The output coupling and feedback gain is derived on the basis of the Lyapunov theory and the linear matrix inequality (LMI) technique. Some numerical examples are given to demonstrate the effectiveness of the main results.

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1. Introduction

During the past 20 years, many famous chaotic systems such as Lorenz system [1], Chua’s circuit [2], Chen system [3], Rössler system [4] and so on have been proposed and their complex behaviors have also been widely studied. Pecora and Carroll [5] first introduced a synchronization methodology for two unidirectional coupled chaotic systems. After that, many efficient schemes and techniques for controlling chaos synchronization such as adaptive control [6], fuzzy control [7], digital control [8], state feedback control [9], sampled driving signals [10], time-delay feedback control [11], and observer-based approach [12], etc have been developed. Furthermore, some attractive results of synchronization and applications in secure communication for a class of chaotic systems have been reported [13–16].

For dynamic systems, the effect of time-delay frequently causes system instability and decreases system performance. The stability analysis and control design of time-delay systems has been of great interest to many scientists and engineers over the past years. For chaotic systems with time-delay, several works have proposed the problem for various chaotic systems in the literature [17–21]. In Refs. [17,18], the design method of guaranteed control for stabilizing time-delay chaotic systems has been proposed. In Ref. [19], a time-delay master–slave chaos synchronization of two coupled systems using the unidirectional linear error feedback scheme has been developed. Controlling chaos of the Chen–Lee system with multiple time-delays has been studied in Ref. [20]. Based on the...
Lyapunov exponent and the Galerkin projection technique, the stability and chaos control of multiple time-delays Rössler system was analyzed in Ref. [21]. In recent years, the $H_\infty$ control has been widely applied to stabilize a class of uncertain time-delay systems dealing with external noise and disturbance in Refs. [22–25]. The $H_\infty$ control was also proposed to reduce the effect of the noise/disturbance input to the overall system error state or the regulated output to within a prescribed level [26]. More recently, the $H_\infty$ synchronization problem for a general class of chaotic systems without time-delays in the state had been developed in Ref. [27]. However, to the best of our knowledge, the $H_\infty$ control problem for a class of chaotic systems with multiple time-delays had not been considered in the literature. The purpose of this study is to design a robust output coupling and feedback scheme, which is not only to asymptotically synchronize the master and the slave systems with multiple time-delays but also to guarantee the attenuation performance of noise perturbation within a prescribed index $\gamma$. Two illustrative examples, the Rössler and the Chen chaotic systems with multiple time-delays are presented to demonstrate the effectiveness of the proposed approach.

Throughout this paper, $I$ denotes the identity matrix of appropriate dimensions. For a real matrix $A$, its transpose and spectral norm are denoted by $A^T$ and $\|A\|$, respectively. $Q = Q^T > 0$ ($Q = Q^T < 0$) implies that $Q$ is a symmetric positive (negative) definite matrix. The notation $\ast$ in symmetric block matrices or long matrix expressions throughout the paper represents an ellipsis for terms that are induced by symmetry, e.g. $\begin{bmatrix} B & C \\ D & E \end{bmatrix}$. For a vector $x$, $\|x(t)\|$ means the Euclidean vector norm at time $t$, while $\|x\|_2 := \sqrt{\int_0^\infty \|x(t)\|^2 dt}$. If $\|x\|_2 < \infty$, then $x(t) \in L_2[0, \infty)$ where $L_2[0, \infty)$ stands for the space of square integral functions on $[0, \infty)$.

2. Main result

Consider the master and slave chaotic systems with multiple time-delays described by the following differential Eqs. (1a) and (1b), respectively.

\[
\begin{align*}
\dot{x}_m(t) &= A_x x_m(t) + \sum_{i=1}^N A_i x_m(t - \tau_i) + B g(x_m(t), y_m(t)) + d \\
y_m(t) &= C x_m(t)
\end{align*}
\tag{1a}
\]

and

\[
\begin{align*}
\dot{x}_s(t) &= A_x x_s(t) + \sum_{i=1}^N A_i x_s(t - \tau_i) + B g(x_s(t), y_m(t)) + d + D w(t) + L (y_m(t) - y_s(t)) \\
y_s(t) &= C x_s(t),
\end{align*}
\tag{1b}
\]

where $x_m \in \mathbb{R}^n$ and $x_s \in \mathbb{R}^n$ are the master system’s state and slave system’s state, respectively. $w \in \mathbb{R}^l$ is the external noise perturbation, $d$ denotes the constant input, and $B$, $C$, $D$ are constant matrices with appropriate dimensions. The parameter $L$ is the output coupling and feedback gain. The synchronization error is defined as $e(t) = [x_m - x_s]^T$; therefore, the dynamics of synchronization error between the master–slave systems given in Eqs. (1a) and (1b) can be described by:

\[
\dot{e}(t) = A e(t) + \sum_{i=1}^N A_i e(t - \tau_i) + B [g(x_m(t), y_m(t)) - g(x_s(t), y_m(t))] - D w(t) - L (y_m(t) - y_s(t)),
\tag{2}
\]

where $g(x_m(t), y_m(t))$ and $g(x_s(t), y_m(t))$ satisfy the Lipschitz condition i.e., there exists $\bar{\delta} > 0$ such that $\|g(x_m(t), y_m(t)) - g(x_s(t), y_m(t))\| \leq \bar{\delta} \|x_m(t) - x_s(t)\|$.

This paper aims at designing an output coupling and feedback gain $L$ to not only asymptotically synchronize between the master and the slave systems but also to guarantee a prescribed performance of noise perturbation attenuation $\gamma$. Before presenting the main result, we introduce the following definition.

Definition. For the synchronization error system (2), it is said to have the $H_\infty$ synchronization with the noise perturbation attenuation $\gamma$ if the following conditions are satisfied.
With \( w(t) = 0 \), the dynamics error system (2) is asymptotically stable.

Given a desired positive scalar \( \gamma \) and under zero initial conditions, the following performance index is satisfied:

\[
J = \int_0^\infty \left[ e^T(t)e(t) - \gamma^2 w^T(t)w(t) \right] dt \leq 0.
\]  

(3)

In other words, this paper is to determine the gain \( L \) such that the asymptotic synchroniztion between the master and the slave systems is ensured and the performance index (3) is satisfied, i.e., \( \sup_{w(t) \neq 0, w(t) \in L^2[0, \infty)} \| e(t) \|_2^2 \leq \gamma \).

**Theorem 1.** Consider the synchronization error system (2) with multiple time-delays and noise perturbation input. Given a constant \( \gamma > 0 \), if there exist symmetric positive definite matrices \( P, R_i, i = 1, 2, \ldots, N \), and a matrix \( F \) such that the following LMI condition holds:

\[
\Xi = \begin{bmatrix}
\Xi_{11} & PA_1 & PA_2 & \cdots & PA_N & -P & D & PB \\
* & -R_1 & 0 & 0 & 0 & 0 \\
* & * & -R_2 & 0 & 0 & 0 \\
* & * & * & \ddots & 0 & 0 & 0 \\
* & * & * & * & -R_N & 0 & 0 \\
* & * & * & * & * & -\gamma^2 & 0 \\
* & * & * & * & * & * & -I \\
\end{bmatrix} < 0,
\]

where \( \Xi_{11} = A^TP + PA - C^TF^T - FC + \sum_{i=1}^N R_i + \delta^2 \cdot I + I \).

Then, the system (2) is \( H_\infty \) synchronized with the noise perturbation attenuation \( \gamma \) by the output coupling and feedback gain \( L = P^{-1}F \).

**Proof.** Define the Lyapunov function

\[
V(t) = e^T(t)Pe(t) + \sum_{i=1}^N \int_{t-\tau_i}^t e^T(\beta)R_i e(\beta) d\beta,
\]

(5)

where \( P = P^T > 0 \) and \( R_i = R^T_i > 0, i = 1, 2, \ldots, N \). It is obviously shown that \( V(t) \) is a positive definite function for all \( t \geq 0 \). Evaluating the time derivative of \( V(t) \) along the trajectory given in Eq. (2) gives:

\[
\dot{V}(t) = e^T(t)(A^TP + PA - C^TL^TP - PLC)e(t) + 2 \sum_{i=1}^N e^T(t)PA_ie(t - \tau_i) - 2e^T(t)PDw(t) + (g(x_m(t), y_m(t)) - g(x_s(t), y_m(t)))^TB^TPe + e^TPBe + e^TPB(g(x_m(t), y_m(t)) - g(x_s(t), y_m(t)))
\]

\[
+ \sum_{i=1}^N e^T(t)R_i e(t) - \sum_{i=1}^N e^T(t - \tau_i)R_i e(t - \tau_i).
\]

By the fact \( x^T y + y^T x \leq x^T x + y^T y \) and the Lipschitz condition, we obtain

\[
\dot{V}(t) \leq e^T(t)(A^TP + PA - C^TL^TP - PLC)e(t) + 2 \sum_{i=1}^N e^T(t)PA_ie(t - \tau_i) - 2e^T(t)PDw(t)
\]

\[
+ (g(x_m(t), y_m(t)) - g(x_s(t), y_m(t)))^T(g(x_m(t), y_m(t)) - g(x_s(t), y_m(t)))
\]

\[
+ e^TPBB^TPe + \sum_{i=1}^N e^T(t)R_i e(t) - \sum_{i=1}^N e^T(t - \tau_i)R_i e(t - \tau_i).
\]

(6)
under the output coupling and feedback from 0 to is equivalent
is synchronized with noise perturbation attenuation

\[ y(t) = \sum_{i=1}^{N} P A_i e(t - \tau_i) \]

Define a functional \( J(e(t), w(t)) \) as follows:

\[ J(e(t), w(t)) = \dot{V}(t) + e^T(t) e(t) - \gamma^2 w^T(t) w(t). \]  

Substituting (6) into (7) yields

\[ J(e(t), w(t)) \leq \phi^T \Omega \phi, \]

where \( \phi = [e^T(t) e^T(t - \tau_1) e^T(t - \tau_2) \cdots e^T(t - \tau_N) w^T(t)] \),

\[ \Omega = \begin{bmatrix} \Omega_{11} & PA_1 & PA_2 & \cdots & PA_N & -PD \\ * & -R_1 & 0 & 0 & 0 & 0 \\ * & * & -R_2 & 0 & 0 & 0 \\ * & * & * & \ddots & 0 & 0 \\ * & * & * & * & -R_N & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix}, \]

\[ \Omega_{11} = A^T P + PA - C^T L^T P - PLC + \sum_{i=1}^{N} R_i + \bar{\delta}^2 I + I + PBB^T P. \]

Let \( F = PL \), then we have

\[ \tilde{\Omega} = \begin{bmatrix} \tilde{\Omega}_{11} & PA_1 & PA_2 & \cdots & A_N & -PD \\ * & -R_1 & 0 & 0 & 0 & 0 \\ * & * & -R_2 & 0 & 0 & 0 \\ * & * & * & \ddots & 0 & 0 \\ * & * & * & * & -R_N & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix}, \]

where \( \tilde{\Omega}_{11} = A^T P + PA - F^T P - FC + \sum_{i=1}^{N} R_i + \bar{\delta}^2 I + I + PBB^T P. \)

If there exist matrices \( P, R_i, i = 1, 2, \ldots, N \), and \( F \) such that the condition \( \tilde{\Omega} < 0 \) holds, then

\[ \dot{V}(t)|_{w(t)=0} \leq e^T(t) \cdot \tilde{\Omega} \cdot e(t) < 0, \quad \text{for all } e(t) \neq 0. \]  

Based on the Lyapunov stability theory, the synchronization error system (2) under the output coupling and feedback gain \( L = P^{-1} F \) and \( w(t) = 0 \) is asymptotically stable. By Schur complement [28], the LMI \( \Xi < 0 \) in (4) is equivalent to \( \Omega < 0 \) in (9). Furthermore, the condition of \( \tilde{\Omega} < 0 \) is also equivalent to that of \( \Omega < 0 \). Integrating the function in (7) from 0 to \( \infty \) and by (8)--(10), we have

\[ V(\infty) - V(\phi) + \int_{0}^{\infty} \left[ \|e(t)\|^2_2 - \gamma^2 \cdot \|w(t)\|^2_2 \right] dt \leq 0. \]

With zero initial condition \( (\phi = 0) \), we have

\[ \int_{0}^{\infty} \left[ \|e(t)\|^2_2 - \gamma^2 \cdot \|w(t)\|^2_2 \right] dt \leq 0. \]

Consequently, the system (2) is synchronized with noise perturbation attenuation \( \gamma \) by the output coupling and feedback gain \( L = P^{-1} F \). \( \square \)
3. Examples and simulation results

To demonstrate the validity of the proposed synchronization approach, we consider the following two noise-perturbed chaotic systems with multiple time-delays in this section.

**Example 1.** Consider the noise-perturbed chaotic Rössler system with two time-delays as follows [21]:

\[
\dot{x}_1 = -x_2 - x_3 + a_1 x_1(t - \tau_1) + a_2 x_1(t - \tau_2) \\
\dot{x}_2 = x_1 + z x_2 \\
\dot{x}_3 = b + x_3 x_1 - c x_3.
\]

For instance, the parameters \(a_1 = 0.2, a_2 = 0.4, z = 0.25, b = 0.3, c = 5, \tau_1 = 1, \tau_2 = 2\). The chaotic behavior of Rössler system with two time-delays is shown in Fig. 1. In order to accomplish synchronization, the master and slave chaotic Rössler systems are given by

\[
\dot{x}_m(t) = A x_m(t) + \sum_{i=1}^{N} A_i x_m(t - \tau_i) + B g(x_m(t), y_m(t)) + d \\
= \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.25 & 0 \\ 0 & 0 & -5 \end{bmatrix} x_m(t) + \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_m(t - 1) + \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_m(t - 2) \\
+ \begin{bmatrix} 0 \\ 0 \\ x_{m1} x_{m3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.3 \end{bmatrix}
\]

\[
y_m(t) = C x_m(t) \\
= \begin{bmatrix} 0.1 & 0 & 0 \end{bmatrix} x_m(t)
\]

and

\[
\dot{x}_s(t) = A x_s(t) + \sum_{i=1}^{N} A_i x_s(t - \tau_i) + B g(x_s(t), y_m(t)) + d + D w(t) + L (y_m - y_s) \\
= \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.25 & 0 \\ 0 & 0 & -5 \end{bmatrix} x_s(t) + \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_s(t - 1) + \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_s(t - 2) \\
+ \begin{bmatrix} 0 \\ 0 \\ x_{s1} y_m \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.3 \end{bmatrix} w(t) + \begin{bmatrix} 0.7 & 0.8 \end{bmatrix} (y_m - y_s)
\]

\[
y_s(t) = C x_s(t) \\
= \begin{bmatrix} 0.1 & 0 & 0 \end{bmatrix} x_s(t).
\]

The Lipschitz constant is selected as \(\tilde{\delta} = 0.1 \sup_{t \geq 0} |y_m(t)| = 1\). Furthermore, a time-limited Gaussian noise with mean 0 and variance 1 (Fig. 2) is imposed on the slave system.

By applying the conditions in Theorem 1 and the noise perturbation attenuation \(\gamma = 0.8\), we can obtain the following matrices:

\[
P = \begin{bmatrix} 12.6693 & 2.8397 & -0.9812 \\ 2.8397 & 0.9277 & -0.7149 \\ -0.9812 & -0.7149 & 1.9979 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 11.8512 & 0 & 0 \\ 0 & 0.4549 & 0.0761 \\ 0 & 0.0761 & 1.0234 \end{bmatrix},
\]

\[
R_2 = \begin{bmatrix} 15.0075 & 0 & 0 \\ 0 & 0.4549 & 0.0761 \\ 0 & 0.0761 & 1.0234 \end{bmatrix}, \quad F = \begin{bmatrix} 446.4527 \\ -22.6820 \\ -58.6930 \end{bmatrix}.
\]
Fig. 1. Chaotic behavior of Rössler system with multiple delays.

and the output coupling and feedback gain \( L \) is then given by

\[
L = P^{-1}F = \begin{bmatrix} 195.1066 \\ -787.6767 \\ -215.3970 \end{bmatrix}. \tag{11}
\]

Applying the output coupling and feedback gain (11) without disturbance signal, the synchronization error between the master system and slave system with initial conditions \( x_m = [0.5 \ 0.2 \ 0.5]^T \) and \( x_s = [0.01 \ -0.5 \ 0.1]^T \), respectively, is shown in Fig. 3. It shows that the synchronization error converges to zero. Fig. 4 shows that the effect of the noise perturbation \( w(t) \) on the synchronization error dynamics \( \|e(t)\| \) is reduced to within a prescribed level \( \gamma = 0.8 \) by the output coupling and feedback gain (11).

**Example 2.** Consider a noise-perturbed chaotic Chen system with two time-delays as follows:

\[
\begin{align*}
\dot{x}_1 &= m(x_2 - x_1) + a_1 x_1(t - \tau_1) + a_2 x_1(t - \tau_2) \\
\dot{x}_2 &= -x_1 x_3 + bx_2 + a_3 x_2(t - \tau_1) \\
\dot{x}_3 &= x_1 x_3 - cx_3 + z + a_4 x_2(t - \tau_2).
\end{align*}
\]

For instance, the parameters \( a_1 = 1.4, a_2 = 0.4, a_3 = a_4 = 0.4, b = 20, c = 3, z = -300, m = 36, \tau_1 = 1, \tau_2 = 2 \).

The chaotic behavior of Chen system with two time-delays is shown in Fig. 5. In order to accomplish synchronization, the master and slave chaotic Chen systems are given by

\[
\dot{x}_m(t) = Ax_m(t) + \sum_{i=1}^{N} A_i x_m(t - \tau_i) + Bg(x_m(t), y_m(t)) + d
\]
Fig. 3. Time responses of synchronization error of Rössler system without noise signal \( w(t) \).

\[
\begin{align*}
\dot{y}_m(t) &= C x_m(t) \\
&= \begin{bmatrix} 0.01 & 0 & 0 \end{bmatrix} x_m(t)
\end{align*}
\]

and

\[
\dot{x}_s(t) = A x_s(t) + \sum_{i=1}^{N} A_i x_s(t - \tau_i) + B g(x_s(t), y_m(t)) + d + D w(t) + L (y_m - y_s)
\]

\[
= \begin{bmatrix} -36 & 36 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & -3 \end{bmatrix} x_s(t) + \begin{bmatrix} 1.4 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_s(t - 1) + \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0 & 0 \end{bmatrix} x_s(t - 2)
\]
Fig. 5. Chaotic behavior of Chen system with two time-delays.

Fig. 6. Time responses of synchronization error of Chen system without noise signal $w(t)$

$$y_s(t) = C x_s(t)$$

$$= \begin{bmatrix} 0.01 & 0 & 0 \end{bmatrix} x_s(t).$$

The Lipschitz constant is chosen as $\bar{\delta} = 0.01 \sup_{t \geq 0} |y_m(t)| = 0.65$. Furthermore, a Gaussian noise with mean 0 and variance 1 (Fig. 2) is imposed on the slave system.

By applying the conditions in Theorem 1 and the disturbance attenuation $\gamma = 0.7$, we can obtain the following matrices:

$$P = \begin{bmatrix} 0.7684 & -0.3941 & -0.0378 \\ -0.3941 & 0.5252 & 0.0641 \\ -0.0378 & 0.0641 & 0.8936 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 2.1402 & 0.0007 & -0.0659 \\ 0.0007 & 1.8395 & 0.0793 \\ -0.0659 & 0.0793 & 0.6428 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 1.9288 & -0.0176 & 0.0003 \\ -0.0176 & 1.8555 & 0.0768 \\ 0.0003 & 0.0768 & 0.4861 \end{bmatrix}, \quad F = \begin{bmatrix} -2.3482 \\ 3.3598 \\ 0.2299 \end{bmatrix} \times 10^3.$$
and the output coupling and feedback gain is then given by

\[ L = P^{-1}F = \begin{bmatrix} 0.3697 \\ 6.6999 \\ -0.2078 \end{bmatrix} \times 10^3. \tag{12} \]

Applying (12) without disturbance signal, the synchronization error between the master system and slave system with initial conditions \( x_m = [0.5 \ 0.2 \ 0.5]^T \) and \( x_s = [-0.2 \ -0.3 \ -0.5]^T \), respectively, is shown in Fig. 6. It shows that the synchronization error converges to zero. Fig. 7 shows that the effect of the disturbance input \( w(t) \) on the error dynamics \( \| e(t) \| \) has been reduced to within a prescribed level \( \gamma = 0.7 \) by the output feedback control gain (12).

4. Conclusion

An output coupling and feedback scheme for \( H_\infty \) synchronization with noise perturbation attenuation \( \gamma \) of a class of noise-perturbed chaotic systems with multiple time-delays has been studied. Based on the Lyapunov theory and the LMI optimization technique, the output coupling and feedback gain for the slave systems has been derived not only to guarantee the asymptotic synchronization but also to ensure a prescribed noise perturbation attenuation performance. Finally, we have showed the effectiveness of the proposed scheme through numerical simulations of two famous Rössler and Chen systems with two time-delays.

References


