$u''(x) = \phi(x)$

$u(0) = u(1) = 0$

$\int_0^1 G L u \, d\xi = [\ \ ]_0^1 + \int_0^1 u L^* G d$

$\int_0^1 G \phi \, d\xi = G u' - u G_\xi \Big|_0^1 + \int_0^1 u G$

$\int_0^1 G(\xi,x)\phi(\xi)\,d\xi = \underline{G(1,x)} u'(1) - u(1)$
$G_\xi(1,x) - \underline{G(0,x)} u'(0) + u(0) G_\xi(0,x)$
$+ \int_0^1 u(\xi) L^* G \, d\xi$

Chapter II Green's Function Method

II-1. Introduction

Consider a b.v.p.

Green function 積分方程之特殊形式 ⑴ 式⑴之[illegible] 非齊性解 B.C. [illegible] 邊界 [illegible] 解

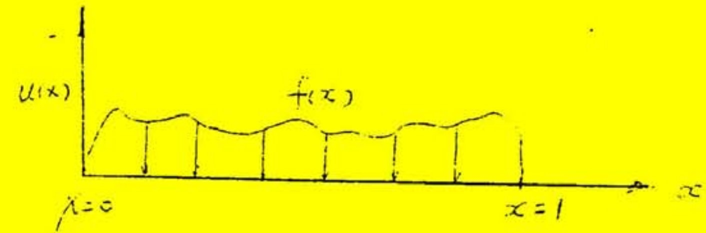
$$L[u] = [p(x)u']' + q(x)u(x) = f(x) \qquad a < x < b$$

with *b.c*'s ; u(a)=u(b)=0

we wish to show the existence of a so-called Green's fuc $G(\xi, x)$ for this prob. such that the sol. is

$$\underbrace{u(x) = \int_a^b G(x,\xi) f(\xi) d\xi} \quad -----(1)$$

Example: a uniform stretched string



$$L[u] = u'' = f(x)$$

$$u(0) = u(1) = 0$$

Let's evaluate the integral

$$\begin{aligned}\int_0^1 G(\xi,x) f(\xi) d\xi &= \int_0^1 G(\xi,x) u_{\xi\xi} d\xi \\ &= \left[G(\xi,x)u_\xi - uG_\xi(\xi,x)\right]_0^1 - \int_0^1 u(\xi) G_{\xi\xi}(\xi,x) d\xi \\ &= \left[G(1,x)u_\xi(1) - u(1)G_\xi(1,x)\right] - \left[G(0,x)u_\xi(0) - u(0)G_\xi(0,x)\right] \\ &\quad - \int_0^1 u(\xi) G_{\xi\xi}(\xi,x) d\xi\end{aligned}$$

If we choose $G(\xi, x)$ clearly, this eq. can provide us with the sol. to an original prob. Specifically, if we require that

$$L^*G = G_{\xi\xi}(\xi, x) = \delta(\xi - x)\ ,\ G(0, x) = G(1, x) = 0 \quad ----(2)$$

Then our solution is

$$\int_0^1 u(\xi)\,\delta(\xi - x)\,d\xi = u(x)$$

$$u(x) = \int_0^1 G(\xi, x) f(\xi) d\xi \quad ----(3)$$

Sol. of (2) $\Rightarrow$

$$G_\xi = H(\xi - x) + A$$

$$G = (\xi - x)H(\xi - x) + A\xi + B$$

where $H(\xi - x)$ is the unit step fuc., defined a

$$H(\xi - x) = \begin{cases} 1 & \text{for } \xi \geq x \\ 0 & \text{for } \xi \leq x \end{cases}$$

and we have utilized

$$H'(\xi - x) = \delta(\xi - x)$$

By $b.c.'s \Rightarrow \quad 0 = B$

$$0 = (1 - x) + A + B \Rightarrow A = x - 1$$

$$\text{therefore} \quad G(\xi, x) = (\xi - x)H(\xi - x) + (x - 1)\xi$$
$$= \begin{cases} (x - 1)\xi & \text{for } \xi < x \\ (\xi - 1)x & \text{for } \xi > x \end{cases}$$

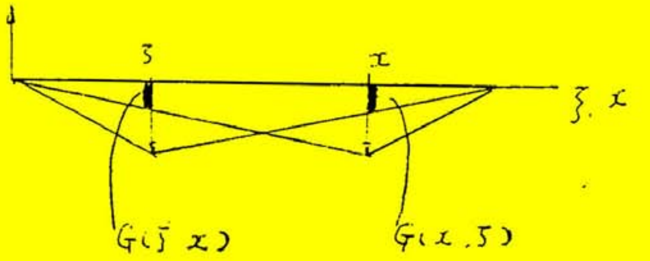
Some Remarks:

(1)since

$$G_{\xi\xi}(\xi, x) = \delta(\xi - x), G(0, x) = G(1, x) = 0$$

$$G_{xx}(x, \xi) = \delta(x - \xi),\ G(0, \xi) = G(1, \xi) = 0$$

physically, $G(\xi, x)$ is the defection at ξ, due to a pt. load of unit strength acting at the pt. $\xi = x$



we observe $G(x, \xi) = G(\xi, x)$

i.e., $G(\xi, x)$ is symmetric 對ξ, x，即ξ, x對調

$\Rightarrow$ the defection at ξ due to a unit load at x is equal to the defection at x due to a unit load at ξ.

(2)Clearly, $G(x, \xi) f(\xi) d\xi$ is the deflection at x due to an incremental load $f(\xi)d\xi$ *at* ξ, so that the solution represents the superposition of the resulting incremental deflections.

from(2)

$$\Rightarrow u = \int_0^1 G(\xi, x) f(\xi) d\xi$$

$$u = \int_0^1 G(x, \xi) f(\xi) d\xi$$

(3)Adjoint *Green'* fuc

Green's fuc. for a system satisfies

$L^* G(\xi, x) = \delta(\xi - x) +$ suitable homo. *b*.c's say BC

The adjoint system is

$(L^*)^* G^*(\xi, x') = L G^*(\xi, x') = \delta(\xi - x') + B.C^*$

If the operation L is self-adjoint, we have

$(Lu, v) = (u, L^* v)$

let $u = G^*, v = G$

$\Rightarrow \int_a^b \delta(\xi - x') G(\xi, x) d\xi = \int_a^b G^*(\xi, x') \delta(\xi - x') d\xi$

$\Rightarrow G(x', x) = G^*(x, x')$

So if the system is self-adjoint $\Rightarrow \underline{G(\xi, x) = G(x, \xi)}$ 驗證 (利用 self-Adjoint 之特性)

II-II. Construction of *Green's* Fuction

（非齊次項）

$$L[u] = [p(x)u'(x)]' + q(x)u(x) = f(x)$$ 物理意義是一些 Loading

$$B_a[u] = \alpha_1 u(a) + \alpha_2 u'(a) = B_a$$

$$B_b[u] = \beta_1 u(b) + \beta_2 u'(b) = B_b$$

Consider

$$\int_a^b G(\xi, x)L\ [u]d\xi$$

$$= \int_a^b G(\xi, x)\{[p(\xi)u'(\xi)]' + q(\xi)u(\xi)\}d\xi$$

$$= [Gpu_\xi - G_\xi pu]_a^b + \int_a^b u(\xi)L\ ^*G(\xi, x)d\xi$$

Consider the boundary term

$$G(b, x)u_\xi(b) - G_\xi(b, x)u(b)$$

$$= \frac{G(b,x)}{\beta_2}\{\beta_2 u_\xi(b) + \beta_1 u(b)[-\frac{\beta_2 G_\xi(b,x)}{\beta_1 G(b,x)}]\}$$

$$= \begin{cases} \frac{1}{\beta_2}G(b, x)\mathcal{B}_b (if\ \ \beta_2 \neq 0, \frac{\beta_2 G_\xi(b,x)}{\beta_1 G(b,x)} = -1) \\ \qquad or\ \ \mathcal{B}_b[G] = \beta_1 G(b, x) + \beta_2 G_\xi(b) = 0 \\ \frac{-1}{\beta_1}G_\xi(b, x)\mathcal{B}_b \quad (if\ \ \beta_2 = 0, G(b, x) = 0) \\ \qquad or\ \ \mathcal{B}_b[G] = \beta_1 G(b, x) = 0 \end{cases}$$

Similarly, $\quad G(a, x)u_\xi(a) - G_\xi(a, x)u(a)$

$$= \frac{1}{\alpha_2}G(a, x)\mathcal{B}_a \quad (if \alpha_2 \neq 0, \mathcal{B}_a[G] = \alpha_1 G(a, x) + \alpha_2 G_\xi(a, x) = 0)$$

$$= \frac{-1}{\alpha_1}G_\xi(a, x)\mathcal{B}_a \quad (if \alpha_2 = 0, \mathcal{B}_a[G] = \alpha_1 G(a, x) = 0)$$

So clearly, if we choose $L^*G(\xi, x) = \delta(\xi - x)$

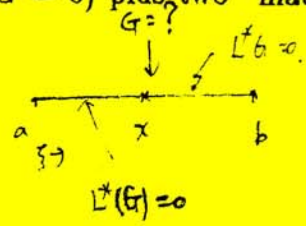
with b.c's: $\mathcal{B}_a[G] = \mathcal{B}_b[G] = 0$

$$\Rightarrow \ u(x) = \int_a^b G(\xi, x)f(\xi)d\xi$$

$$+ \begin{cases} (i)\frac{p(b)}{\beta_2}G(b, x)\mathcal{B}_b - \frac{p(a)}{\alpha_2}G(a, x)\mathcal{B}_a & (\text{if } \alpha_2 \neq 0, \beta_2 \neq 0) \\ (ii)\frac{p(b)}{\beta_2}G(b, x)\mathcal{B}_b + \frac{p(a)}{\alpha_1}G(a, x)\mathcal{B}_a & (\text{if } \alpha_2 \overset{=}{\neq} 0, \beta_2 \doteqdot 0) \\ (iii) - \frac{p(b)}{\beta_1}G_\xi(b, x)\mathcal{B}_b + \frac{p(a)}{\alpha_1}G_\xi(a, x)\mathcal{B}_a & (\text{if } \alpha_2 = 0, \beta_2 = 0) \\ (iv) - \frac{p(b)}{\beta_1}G_\xi(b, x)\mathcal{B}_b - \frac{p(a)}{\alpha_2}G(a, x)\mathcal{B}_a & (\text{if } \alpha_2 \doteqdot 0, \beta_2 \underset{=}{\neq} 0) \end{cases}$$

(A)First Method

Since $\delta(\xi - x) = 0$ for all $\xi \neq x$, it will be conveinent to split the interval into two parts $a < \xi < x$, and $x < \xi < b$ (in each interval $L^*G = 0$) plus two "matching conditions" which blind the two parts suitably at $\xi = x$.



By integration the eq. from "x-0" to "x+0"

$$\int_{x-0}^{x+0} \frac{d}{d\xi}[p(\xi)\frac{dG}{d\xi}] + q(\xi)G(\xi,x)\}d\xi = \int_{x-0}^{x+0} \delta(\xi - x)d\xi$$

$$[p(\xi)\frac{dG(\xi,x)}{d\xi}]_{x-0}^{x+0} + \int_{x-0}^{x+0} q(\xi)\overset{0}{G}(\xi,x)d\xi = 1$$

We demand that $G(\xi,x)$ be a continuous fuc. of ξ at $\xi = x$, then

$$G(x+0,x) = G(x-0,x)$$

1st matching condi.

and since $p(\xi), q(\xi)$ are continuous on[a,b], then the 2nd term drops out, hence the 1st term expresses the 2nd condition.

$$[G_\xi]_{x-0}^{x+0} = \frac{1}{p(x)}$$

Hence ⇒

$L^*G = 0$ $\underset{\text{with}}{\text{m}}$ each interval -------- (1)

$B_a[G(a,x)] = B_b[G(b,x)] = 0$ -------(2)

$G(x+0,x) = G(x-0,x)$ -------- (3)

$\frac{\partial G}{\partial \xi}(x+0,x) - \frac{\partial G}{\partial \xi}(x-0,x) = \frac{1}{p(x)}$ ------ (4)

(B)Second Method:

(1) If $L^*[u] = 0, +B_a[u] = B_b[u] = 0$ has no nontrivial solution. (only have trivial sol. u=0)

Then, consider two fucs $u_1(x), u_2(x)$ which satisfy

$L[u_1] = 0$ $\quad B_a(u_1) = B_b(u_2) = 0$

$L[u_2] = 0$ which are indep. const[illegible] the Wranskian.

$$W(u_1,u_2) = u_1(x)u_2{}'(x) - u_2(x)u_1{}'(x) \neq 0$$

Then the *Green's* fuc is given by

[illegible]

$$G(\xi,x)=\begin{cases}\frac{u_1(\xi)u_2(x)}{p(\xi)w(\xi)} & \text{for } \xi<x\\ \frac{u_1(x)u_2(\xi)}{p(\xi)w(\xi)} & \text{for } \xi>x\end{cases}$$

Verify: It is obvious that (1).(2).(3) are satisfied.

now $\frac{\delta G}{\delta\xi}(x-0,x)=\frac{pwu_1{}'u_2-u_1u_2(pw)'}{(pw)^2}(x)$

$\frac{\delta G}{\delta\xi}(x+0,x)=\frac{pwu_1u_2{}'-u_1u_2(pw)'}{(pw)^2}(x)$

$\Rightarrow \frac{\delta G}{\delta\xi}(x+0,x)-\frac{\delta G}{\delta\xi}(x-0,x)=\frac{1}{(pw)^2}pw(u_1u_2{}'-u_1{}'u_2)$

$=\frac{pw\cdot w}{(pw)^2}=\frac{1}{p(x)}$ (OK) (4)

Ex.: $L[u]=\frac{d}{dx}[k(x)\frac{du}{dx}]=f(x)$

$b.c's:$ $u(0)=u(1)=0$

Sol.:1st find $L^*[u]=[ku']'=0$

$\Rightarrow ku'=c\Rightarrow u(x)=\int\frac{c}{k(x)}dx$

To satisfy the $b.c$'s, $\Rightarrow$ the trivial sol. $u\equiv 0$

2nd, const two linearly indep. fucs

$u_1(x)=\int_0^x c\frac{d\xi}{\kappa(\xi)}$

$u_2(x)=\int_x^1\frac{d\xi}{k(\xi)}$ which satisfy$u_1(0)=u_2(1)=0$

The Wranskian of u_1,u_2

$$\begin{aligned} w&=-\frac{1}{k(x)}u_1(x)-\frac{1}{k(x)}u_2(x)\\ &=-\frac{1}{k}[u_1(x)+u_2(x)]\\ &=-\frac{1}{k(x)}\int_0^1\frac{d\xi}{k(\xi)}\\ &=-\frac{c_1}{k(x)} \end{aligned}$$

$where$ $c_1=\int_0^1\frac{d\xi}{k(\xi)}$

So the *Green*'s fuc. is

$$G(\xi,x)=\begin{cases}-\frac{\int_0^{\xi}\frac{dx'}{k(x')}\int_x^1\frac{dx'}{k(x')}}{c_1} & for\quad \xi<x\\ -\frac{\int_0^{x}\frac{dx'}{k(x')}\int_{\xi}^{1}\frac{dx'}{k(x')}}{c_1} & for\quad \xi>x\end{cases}$$

(2) When $L[u]=0+B_a[u]=B_b[u]=0$ has at least one nontrivial solution, then the "usual" *Green*'s fuc. does not exist.

Ex.: $L[u]=u''+u=f(x)$

$u(0)=u(\pi)=0$

clearly, the homo. eq. $L[u] = 0 + b.c'su(0) = u(\pi) = 0$ has a nontrivial sol. $\sin x$

The *Green's* fuc. satisfies

$$L^*G = G_{\xi\xi} + G = \delta(\xi - x)$$

$$G(0, x) = G(\pi, x) = 0$$

As in the previous procedures, the *Green's* fuc. is in the form

$$G(\xi, x) = \begin{cases} \mathrm{A(x)} \sin\xi & + & B(x)\cos\xi & \xi < x \\ \mathrm{C(x)} \sin\xi & + & D(x)\cos\xi & \xi > x \end{cases}$$

However, we will show $G(\xi, x)$ does not exist for satisfying the $b.c$'s and the matching conditions.

For the $b.c's$ $\Rightarrow$ $B = D = 0$

For G to be continuous at $\xi = x$

$$\Rightarrow A = C$$

For the jump conditon

$$G_\xi|_{x-0}^{x+0} = \frac{1}{p(x)} = 1$$

$$\Rightarrow C\cos x - A\cos x = 1$$

Unfortunately, two unknowns A.C are incompatible! $\Rightarrow$ our "usual" *Green's* fuc. does not exist!

Actually, this situation does not arise very often in practice. Nevertheless, it is well worth discussing not only for the sake of completness, but also some important feature of linear eq. will be brought in the discussion.

II-3 The Generalized (Modified) Green's Functions

For the example discussed above, to get the source of difficulty let $v(\xi)$ be the nontrivial sol. of the homo. system.

$$\Rightarrow \int_0^\pi v(\xi)(G_{\xi\xi}+G)d\xi \quad = \quad \int_0^\pi v(\xi)\delta(\xi-x)d\xi$$

$$[uG_\xi - v_\xi G]_0^\pi + \int_0^\pi G(v_{\xi\xi}+v)d\xi \quad = \quad v(x)$$

$$\Rightarrow \quad 0+0=v(x) \quad \Rightarrow \quad Contradictions!$$

One might say, then, the difficulty is due to the fast that $v(\xi)=\sin\xi$ is not "orhtogonal" to $\delta(\xi-x)$. This provides us with a clue as to how we can patch things up.

Let's require that

$$L^*G = G_{\xi\xi}+G = \delta(\xi-x)+F$$

$$G(0,x)=G(\pi,x)=0$$

where F is so chosen that $\sin\xi$ is orthogonal to $\delta(\xi-x)+F$. We will demonstrate that a suitable general form for F is $F=\alpha v(x)v(\xi)$, the const. α can in fact be choosen so as to obtain the desired orthogonality.

since $\int_0^\pi \nu(\xi)[\delta(\xi-x)+\alpha\nu(x)\nu(\xi)]d\xi$

$$= \qquad \nu(x)[1+\alpha\int_0^\pi \nu^2(\xi)d\xi]=0$$

$$\Rightarrow \alpha = -\frac{1}{\int_0^\pi \nu^2(\xi)d\xi}$$

So we conclude that G satisifies

$$\begin{cases} \mathrm{L}^*G = \ominus\alpha\nu(x)\nu(\xi)+\delta(\xi-x) \\ \mathrm{B}_a[G]=B_b[G]=0 \\ \mathrm{G}(\xi,x)\text{is continuous at}\,\xi=x \\ \mathrm{G}_\xi \quad \text{has jump}\,\frac{1}{p(x)} \quad at \quad \xi=x \end{cases}$$

where $v(\xi)$ is the nontrivial solution of the homo. eq. $+$ $b.c'$ and $\alpha = \frac{1}{\int_a^b v^2(\xi)d\xi}$

If there are two (lin. indep.) nontrivial sols. say $\nu_1(\xi)$ *and* $\nu_2(\xi)$, then we would require that

$L^*G=\delta(\xi-x)+\alpha_1\nu_1(x)\nu_1(\xi)+\alpha_2\nu_2(x)\nu_2(\xi)$ *where* α_1, $\alpha_2$are chosen so that

$$\int_0^\pi \nu_j(\xi)\left[\delta(\xi-x)+\alpha_1\nu_1(x)\alpha_1(\xi)+\alpha_2\nu_2(x)\nu_2(\xi)\right]d\xi=0$$

$$j=1,2$$

I-IV *Green's* Fuction By Eigenfuc Expansion

$$L[u] = f + homo. \quad b.c's$$

where L is a self-adjoint 2nd order *o.d.e.*

Solve the associated homo. prob.

$$L[u] + \lambda u = 0 + \textbf{same homo. } b.c'.$$

and find the eigenvalues λ_n

eigenfucs ϕ_n

Seek the solution in the form

$$u(x) = \sum a_n \phi_n(x) \textbf{which satisifies} b.c's$$

Expand $f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x)$

where $f_n = \frac{(f, \phi_n)}{(\phi_n, \phi_n)}$

Then EQ $\Rightarrow$ $L \sum a_n \phi_n = -\sum a_n \lambda_n \phi_n = \sum f_n \phi_n$

$$\Rightarrow a_n = \frac{-f_n}{\lambda_n}$$

$$\Rightarrow u(x) = -\sum_{n=1}^{\infty} \left(\frac{f_n}{\lambda_n}\right) \phi_n(x)$$

$$= \int_a^b \left\{ -\sum_{n=1}^{\infty} \frac{\phi_n(\xi)\phi_n(x)}{\lambda_n(\phi_n, \phi_n)} \right\} f(\xi) d\xi$$

$$= \int_a^b G(\xi, x) f(\xi) d\xi$$

where $G(\xi, x) = -\sum_{n=1}^{\infty} \frac{\phi_n(\xi)\phi_n(x)}{\lambda_n(\phi_n, \phi_n)}$

Conclusions:

(1) If all the eigenvalues $\lambda_n \neq 0$ **then both the sol. and the** *Green'* **s fuc. exist.**

(2) If one of the eigenvalues, say $\lambda_j = 0$ **then the** *Green's* **fails to exist.**

(i) If $f_j = (f, \phi_j) \neq 0$**, sol. of u fails to exist.**

(ii) If $f_j = 0$**, an infinitely many sols may exist.**

此時 f 與 ϕ_j 正交,

Ex.:

$$u'' = f(x)$$

$$u(0) = u(1) = 0$$

The associated S-L prob. is

$$u'' + \lambda u = 0, u(0) = u(1) = 0$$

we have found $\lambda_n = n^2\pi^2 \quad n = 1, 2, \ldots$

$$\phi_n(x) = \sin n\pi x$$

$$(\phi_n, \phi_n) = \int_0^1 \sin^2 n\pi x dx = \frac{1}{2}$$

so $G(\xi, x) = -\sum_{n=1}^{\infty} \frac{2\sin n\pi\xi \sin n\pi x}{n^2\pi^2} \quad ------(1)$

On the other hand, we find

$$G(\xi, x) = \begin{cases} (\mathbf{x\text{-}1})\xi & \xi < x \\ (\xi - 1)x & \xi > x \end{cases} \quad ------(2)$$

These two forms of G are, of course, in agreement, since (1) is simply the Fourier since series expansion of (2). (2) 的 fourier sin expansion 可得 (1)

II-V *Green's* Function Method of P.D.E.

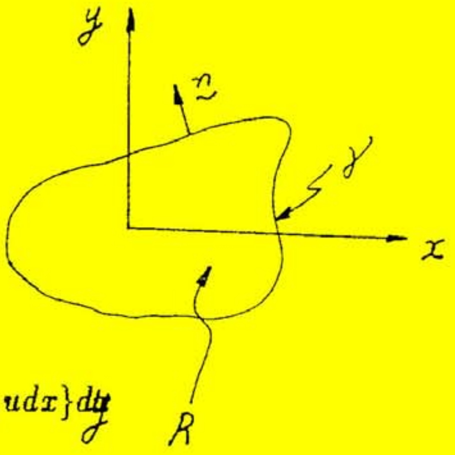
[illegible] 界條件 Domain

Consider

$$L[u] = Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = f(x,y)$$

where A,B,C,... are fucs of x.y only in R. + general form of linear b.c

$$B[u] = \alpha u + \beta u_n = \phi \quad on \quad \gamma$$



From

$$\int vLud\sigma = \quad bndy \quad terms + \int_R uL^* vd\sigma$$

for example

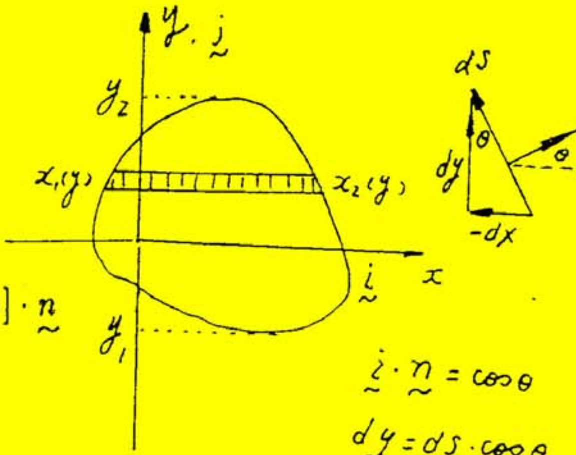
$$\int_R vAu_{xx}dxdy =$$
$$= \int_{y_1}^{y_2}\{\int_{x_1(y)}^{x_2(y)} vA\, u_{xx}dx\}\, dy$$
$$= \int_{y_1}^{y_2}\{[vAu_x - (vA)_x u]_{x_1(y)}^{x_2(y)} + \int_{x_1(y)}^{x_2(y)} (vA)_{xx} u dx\} dy$$
$$+ \oint_\gamma [vAu_x - (vA)_x u]\underset{\sim}{i}\cdot\underset{\sim}{n} ds + \int_R u(vA)_{xx} d\sigma$$

Similar procedures applied to other terms

$$\Rightarrow \int_R vLud\sigma = \oint_\gamma (M\underset{\sim}{i} + N\underset{\sim}{j})\cdot\underset{\sim}{n} ds + \int_R uL^* vd\sigma \quad ------(1)$$

Where $L^*v = (Av)_{xx} + 2(Bv)_{xy} + (Cv)_{yy} - (Dv)_x - (Ev)_y + Fv$

$$M = Avu_x - u(Av)_x + 2v(Bu)_y + Duv$$
$$N = cvu_y - u(cv)_y + 2v(Bu)_x + Euv$$



(1) If $L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ i.e. A=C=1, B=D=E=F=0

so that $M = vu_x - uv_x, N = vu_y - uv_y$

$$\Rightarrow (M\underset{\sim}{i} + N\underset{\sim}{j})\cdot\underset{\sim}{n} = [v(u_x\underset{\sim}{i} + u_y\underset{\sim}{j}) - u(v_x\underset{\sim}{i} + v_y\underset{\sim}{j})]\cdot\underset{\sim}{n}$$
$$= (v\nabla u - u\nabla v)\cdot\underset{\sim}{n} = vu_n - uv_n$$

$\underset{\sim}{i}\cdot\underset{\sim}{n} = \cos\theta$

$dy = ds\cdot\cos\theta$

$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ $= ds\cdot\underset{\sim}{i}\cdot\underset{\sim}{n}$

(1) $\Rightarrow$

$$\int_R vLud\sigma = \oint_\gamma (vu_n - uv_n)ds + \int_R uL^* vd\sigma$$

(2) If $L = k\frac{\partial}{\partial y} - \frac{\partial^2}{\partial x^2}$ Boundary term

Then (1) $\Rightarrow \int_R vLud\sigma = \oint_\gamma[(uv_x - vu_x)\underset{\sim}{i} + xuv\underset{\sim}{j}]\cdot\underset{\sim}{n} ds + \int_R uL^* vd\sigma$

Boundary term : [illegible]

where $L^* = -k \cdot \frac{\partial}{\partial y} - \frac{\partial^2}{\partial x^2}$

(3) If $L = c^2\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$

(1)$\Rightarrow \int_R vLud\sigma = \oint_\gamma[c^2(vu_x - uv_x)\underset{\sim}{i} - (vu_y - uv_y)\underset{\sim}{j}]\cdot\underset{\sim}{n} ds + \int_R uL^* vd\sigma$

where $L^* = L$

In (1),If we choose v to be ours *Green's* fuc G.

Refining that G satisfying

$$L^*G = \delta(\xi - x, \eta - y) \quad over \quad R \quad - - - - - - (2)$$

and subjecting G to b.c on γ which result in the elimination of any "unwelcome" terms in the boundary terms. In general, it is conveinent to seek G into two parts

G = F + g

where F is a particular sol. of (2) which neek not satisfy the b.c and g is a sol. of homo. eq. $L^*G = 0$ such that the combination F+g does satisfy *b.c's* and F is called principal sol, or fundamental sol., or elementary sol., or free-space *Green's* fuc.

Ex.:

$$L^*G = G_{\xi\xi} = \delta(\xi - x)$$
$$G(0, x) = G(1, x) = 0$$

Set $G = F + g$

where $F_{\xi\xi} = \delta(\xi - x) \Rightarrow F = (\xi - x)H(\xi - x)$

$$g_{\xi\xi} = 0 \Rightarrow g = A\xi + B$$
$$\Rightarrow \quad G = (\xi - x)H(\xi - x) + A\xi + B$$

$\xi = 0 \Rightarrow G = 0$ [illegible]

$u(x,y) = \int G f \, d\tau - \oint_\gamma [G u_n - [illegible]] ds$

$G = F + g$

[illegible]

$u = \int_R F f \, d\tau - \oint (F u_n - u F_n) ds =$ [illegible]

II-VI Fundamental Solution

(A)For Laplace operator

$$L^{*}F = \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial y^2}\right)F(\xi,\eta;x,y) = \delta(\xi - x;\eta - y) \quad ------(1)$$

From potential theory, F can be regarded as the potential induced at a fixed point (ξ,η) due to a point mass of unit strength at (x,y). It should therefore be symmetric about the pt (x,y) and will depand only on the radial var.

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2}$$

For $\quad r > 0, \delta(r) = 0$

Eq. (1) $\Rightarrow \nabla^2 F = \frac{1}{r}\frac{d}{dr}(r\frac{dF}{dr}) = 0$

$\quad therefore \quad F = A \quad ln(\ r) + B$

From $\quad \int_{r=\epsilon} \nabla^2 F d\sigma = \int_{r=\epsilon} \delta(r) = 1$

$\quad \Rightarrow \oint_{r=\epsilon} \boxed{\underset{\sim}{n} \cdot \nabla F} ds = 1$

$\quad therefore \quad \oint_{r=\epsilon} \boxed{\frac{dF}{dr}}_{r=\epsilon} ds = 1$

$\quad \Rightarrow \frac{A}{\epsilon}\int_{r=\epsilon} ds = \frac{A}{\epsilon}\cdot 2\pi\epsilon = 1$

$\quad \Rightarrow A = \frac{1}{2\pi}$, B is arb $= 0$

$\quad \Rightarrow F = \frac{1}{2\pi} \quad ln \quad r$

(B)For Diffusion Operator

$$L^{*}F = -\kappa F_{\tau} - F_{\xi\xi} = \delta(\xi - x;\tau - t) \quad ------(2)$$

In this case, no symmetric property is available. Consider if $\quad F^{*} \quad$ were governed by

$$LF^{*} = \kappa F_{\tau}{}^{*} - F_{\xi\xi}{}^{*} = \delta(\xi - x, \tau - t) \quad ------(3)$$

Physically, $LF^{*} = 0$ for all time $\tau < t$, at $\tau = t$, a unit heat source is applied at $\xi = x$, and F^{*} would describe the subsequent diffusion of that heat source for $\tau > t$. Now retruning to (2), we see that the situation is identical except that time is reversed. Thus, our principal sol. will be zero for all $\tau > t$. At heat pulse is applied at $\tau = t$ and F describes the diffusion of this pulse as τ decresases.

Whethere or not this situation is physically possible is beside the point since (2) is only a formal mathematical formulation.

Take F.T. w.r.t. ξ of (2) $\Rightarrow$

$$-\kappa \tilde{F}_{\tau} + \omega^2 \tilde{F} = \delta(\tau - t) e^{i\omega x}$$

We obtain $\tilde{F} = \begin{cases} \mathrm{A}e^{\frac{\omega^2 \tau}{\kappa}} & for \quad \tau > t \\ \mathrm{B}e^{\frac{\omega^2 \tau}{\kappa}} & \tau < t \end{cases}$

and then apply

$\int_{t-0}^{t+0} (-\kappa \tilde{F}_{\tau} + \omega^2 \tilde{F}) d\tau = \int_{t-0}^{t+0} \delta(\tau - t) e^{i\omega x} d\tau$

$\Rightarrow -\kappa A e^{\frac{\omega^2 t}{\kappa}} + \kappa B_e^{\frac{\omega^2 t}{\kappa}} = e^{i\omega x}$

but F and hence $\tilde{F}$ is zero for$\tau > t \Rightarrow A = 0$

therefore $B = \frac{1}{\kappa} e^{\frac{i\omega x - \omega^2 t}{\kappa}}$ $B = \frac{1}{\kappa} e^{\frac{\ }{\kappa}}$

so that $\tilde{F} = \frac{1}{\kappa} H(t - \tau) e^{\frac{i\omega x - \omega^2 (t - \tau)}{\kappa}}$

and then

$F = \frac{H(t-\tau)}{2\pi\kappa} \int_{-\infty}^{\infty} e^{\frac{i\omega(x-\xi) - \omega^2 (t-\tau)}{\kappa}} d\omega$

$= \frac{H(t-\tau) e^{\frac{-\kappa(\xi - x)^2}{4(t-\tau)}}}{[4\pi\kappa(t-\tau)]^{\frac{1}{2}}}$

(C) wave operator

$L^* F = c^2 F_{\xi\xi} - F_{\tau\tau} = \delta(\xi - x; \tau - t)$ ------ (4)

For all $\tau < t$, F is identically zero; at $\tau = t$, a unit impulsive force is applied to the string at $\xi = x$ and F describes its subsequent disp.

Let $F(\xi, \tau; x, t) = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega f(k, \omega) e^{ik(\xi - x)} e^{-i\omega(\tau - t)}$

Take $\underbrace{\text{(double}}$ F.T. of (4) $\Rightarrow$

$$(-c^2 k^2 + \omega^2) f(k, \omega) = \frac{1}{(2\pi)^2}$$

$$\Rightarrow f(k, \omega) = \frac{1}{4\pi^2} \frac{1}{\omega^2 - c^2 k}$$

so that

$$F(\xi, \tau; x, t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk e^{ik(\xi - x)} \int_{-\infty}^{\infty} d\omega \frac{1}{\omega^2 - c^2 k^2} e^{-i\omega(\tau - t)}$$

Consider the integral $I = \int_{-\infty}^{\infty} \frac{e^{-i\omega(\tau - t)}}{\omega^2 - c^2k^2} d\omega$

which has singularities at $\omega = \pm \; ck$

To evaluate the integral, there are two possible contours as shown

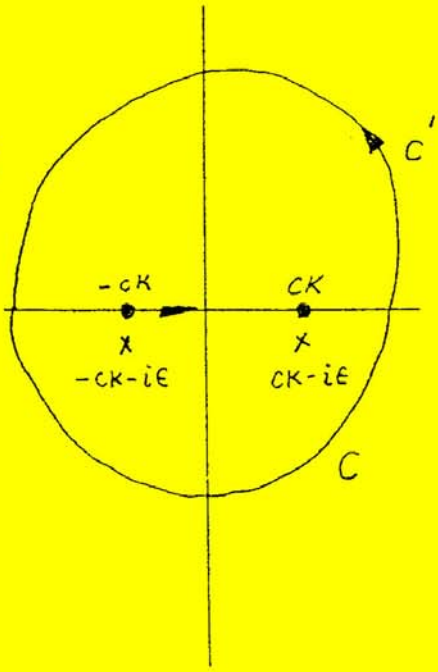
(i) for $\tau > t$, F is diverging wave so $R_e(-i\omega) = \omega_I < 0$

hence the proper contaur is along c

(ii) for $\tau < t, F \equiv 0, \Rightarrow$ the contaur is along c'

In other to make $F \equiv 0 \quad \tau < t,$ we must imagine that the poles at $\omega = \pm ck$ must be displaced below the real axis. Thus the integral along c' will vanish and the integral along c will make contribution.

so $I = \int_{-\infty}^{\infty} \frac{e^{-i\omega T}}{(\omega + i\epsilon)^2 - c^2k^2} d\omega \quad where \quad T = \tau - t$



By Cauchy I.theorem

$$I = -2\pi i [Res \; (\; \omega = -ck - i\epsilon) + Res \; (\; \omega = ck - i\epsilon)]$$

$$= -2\pi i [\frac{e^{ickT}}{-2ck} + \frac{e^{-ickT}}{2ck}]$$

$$= -2\pi \frac{1}{ck} \sin cKT$$

Hence $F = -\frac{1}{2\pi c} \int_{-\infty}^{\infty} dk e^{ik(\xi - x)} \frac{\sin ck(\tau - t)}{k}$

$$= -\frac{1}{\pi c} \int_0^{\infty} dk \cos k(\xi - x) \frac{\sin ck(\tau - t)}{k}$$

since $\int_0^{\infty} \frac{\cos nk \sin mk}{k} dk$

$$= \begin{cases} \pi/2 & for \quad m > |n| > 0 \\ 0 & for \quad |n| \geq m > 0 \end{cases}$$

$therefore \quad F = -\frac{1}{\pi c} \frac{\pi}{2} H[c(\tau - t) - |\xi - x|]$

$$= -\frac{1}{2c} H[(\tau - t) - |\xi - x|]$$

$therefore \quad F = \begin{cases} -1/2c & when \; |\xi - x| < c(\tau - t) \\ 0 & \text{when} \quad |\xi - x| > c(\tau - t) \end{cases}$ ——(5)

Another Principal Solution

We will prove that if some fuc. $f(\xi - x, \tau - t)$ is sol. of (4), then so is $f(\xi - x, t - \tau)$

proof:

Let $\tau - t = t - \tau'$

now since $f(\xi - x, \tau - t) \quad satisfies \quad (4)$

$$\Rightarrow c^2 f_{\xi\xi}(\xi - x, t - \tau') - f_{\tau\tau}(\xi - x, t - \tau') = \delta(\xi - x, t - \tau')$$

But $f_{\tau\tau}(\xi - x, t - \tau') = f_{\tau'\tau'}(\xi - x, t - \tau')$

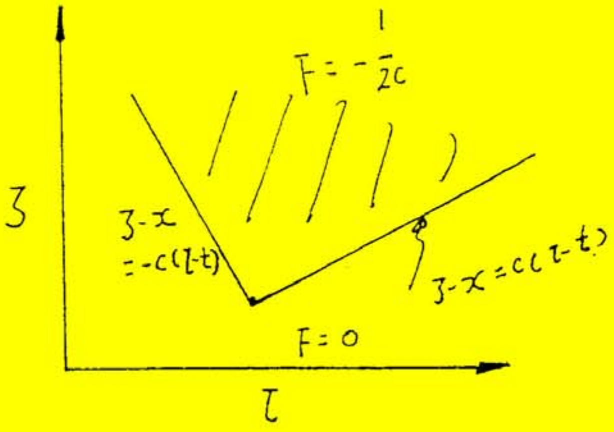
and $\delta(\xi - x, t - \tau') = \delta(\xi - x, \tau' - t)$

so that $c^2 f_{\xi\xi}(\xi - x, t - \tau') - f_{\tau'\tau'}(\xi - x, t - \tau') = \delta(\xi - x, \tau' - t)$

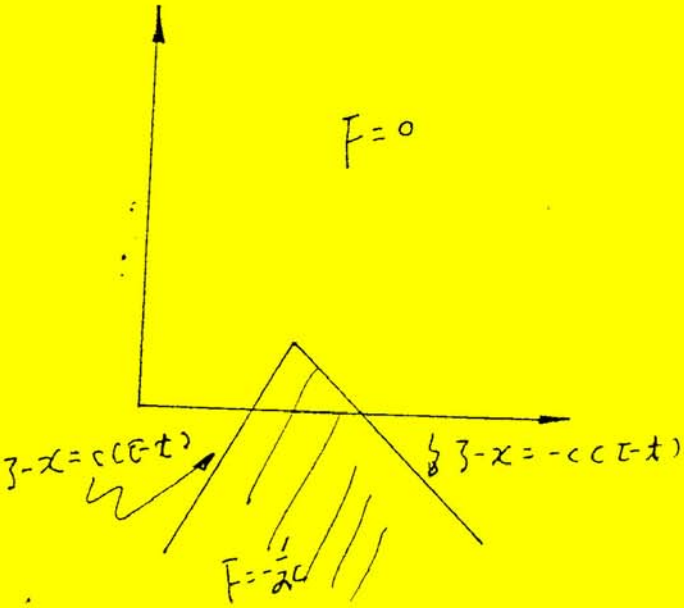
$\Rightarrow c^2 f_{\xi\xi}(\xi - x, t - \tau) - f_{\tau\tau}(\xi - x, t - \tau) = \delta(\xi - x, \tau - t)$ Q.E.D.

$\Rightarrow$ The replacing of $\tau - t$ *in* (5) *by* $t - \tau$, $\Rightarrow$

$$F = \begin{cases} \frac{-1}{2c} & \text{for } |\xi - x| < c(t-\tau) \\ 0 & \text{for } |\xi - x| > c(t - \tau) \end{cases}$$



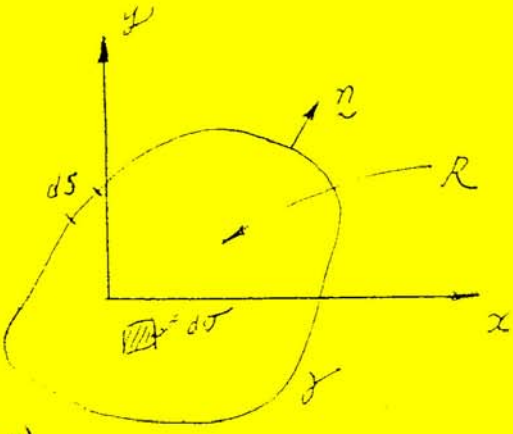
source propagate forward



propagation backward.

II-VII Green's Function Method for the Laplace Operator

(A)2-dim



$$Lu = \nabla^2 u = \rho(x,y) \quad in \quad R.$$

$$B.C.: B[u] = \alpha u + \beta u_n = f \quad on \quad \gamma$$

from $\quad \int_R vLud\sigma = \oint_\gamma (vu_n - uv_n)ds + \int_R uL^* vd\sigma$

If we choose V=G, where G satisfies

$$L^*G = \nabla^2 G = \delta(\xi - x)\delta(\eta - y)$$

then $\quad u(x,y) = \int_R G(\xi,\eta;x,y)\rho(\xi\cdot\eta)d\xi d\eta - \oint_\gamma (Gu_n - uG_n)ds \quad ------(1)$

(i)

If $\quad \beta \neq 0 \Rightarrow Gu_n - uG_n = G(\frac{f-\alpha u}{\beta}) - uG_n$

$$= G(\tfrac{f}{\beta}) - \tfrac{\alpha G + \beta G_n}{\beta} u$$

Hence we require that G satisfies

$$B[G] = \alpha G + \beta G_n = 0 \quad on \quad \gamma \quad -----(2)$$

and solution becomes (1) $\Rightarrow$

$$u(x,y) = -\oint_\gamma \tfrac{1}{\beta} Gfds + \int_R G\rho d\sigma \quad ------(3)$$

which is applied to Neumann prob if $\alpha = 0$.

(ii)

If $\quad \alpha \neq 0 \Rightarrow Gu_n - uG_n = Gu_n - \frac{1}{\alpha}(f - \beta u_n)Gn$

$$= \tfrac{\alpha G + \beta G_n}{\alpha} u_n - \tfrac{f}{\alpha} G_n$$

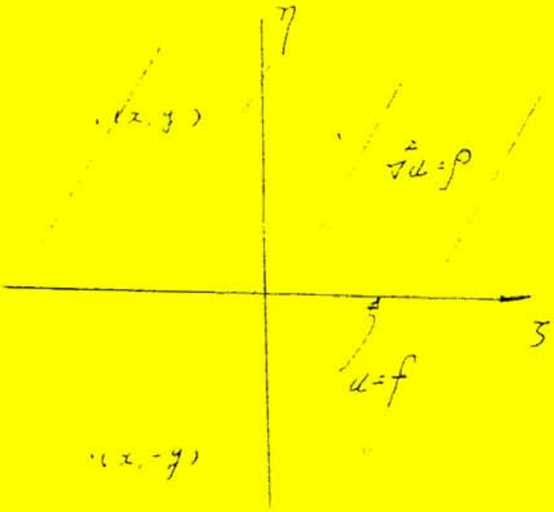
we require G satisfies $\quad B[G] = 0 \quad on \quad \gamma$

Then (1)

$$u(x,y) = \int_R G\rho d\sigma + \oint \tfrac{1}{\alpha} G_n fds$$

which is applied to Dirichlet prob. if $\beta = 0$

Ex.: potential in half-plane



The *Green*'s fuc satisfies

$$\nabla^2 G = \delta(\xi - x)\delta(\eta - y)$$

$$G = 0 \quad on \quad \eta = 0$$

The fundamental sol. for ∇^2 is

$$F = \tfrac{1}{2\pi}\ln r = \tfrac{1}{2\pi}\ln\left[(\xi - x)^2 + (\eta - y)^2\right]^{1/2}$$

$$= \tfrac{1}{4\pi}\ln[(\xi - x)^2 + (\eta - y)^2]$$

Seek G = F + g

with $\nabla^2 g = 0 \quad and \quad g = -F \quad on \quad \gamma$

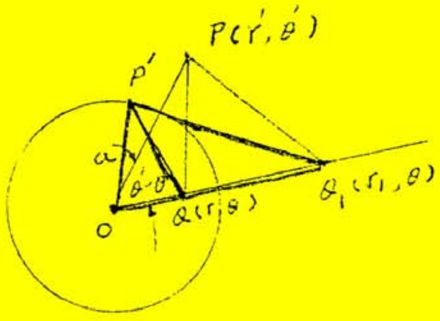
By the method of image, we find that

$$G = \frac{1}{4\pi}\{\ln[(\xi - x)^2 + (\eta - y)^2] - \ln[(\xi - x)^2 + (\eta + y)^2]\}$$

and then

$$G_n|_\gamma = -\tfrac{\partial G}{\partial \eta}|_{\eta=0} = \tfrac{y}{\pi[(\xi - x)^2 + y^2]}$$

$$\Rightarrow \quad u(x, y) = \tfrac{y}{\pi}\int_{-\infty}^{\infty}\tfrac{f(\xi)}{(\xi - x)^2 + y^2}d\xi + \tfrac{1}{4\pi}\int_0^\infty\int_{-\infty}^{\infty}\ln[\tfrac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2}]\rho(\xi, \eta)d\xi d\eta$$



Ex.: Dirichlet prob. for a circle

$$\nabla^2 u = 0 \quad in \quad r = a$$

$$u(a, \theta) = f(\theta)$$

we require $\nabla^2 G = \delta(\mathbf{x}' - \mathbf{x})$

$$G(a, \theta'; r, \theta) = 0$$

The fundamental sol. is $\frac{1}{2\pi}\ln r'$.

A point source is located at $Q(r, \theta)$, we shall seek the potential such that it vanishes at $r' = a$. By image method, the symmetry ⇒ the image source will lie on the ray from the origin to the some at $Q_1(r, \theta)$.

if set $\overline{oQ0Q_1} = a^2 \quad - - - - - - (1)$

let p : any observing pt.

if $p \to p'$ on the boundary we require the potential vanishes.

In $\Delta op'Q, and \Delta oQ_1p'$

from (1) $\Rightarrow \frac{\overline{oQ}}{a} = \frac{a}{\overline{oQ_1}}$

$$i.e. \frac{\overline{oQ}}{\overline{op'}} = \frac{\overline{op'}}{\overline{oQ_1}} \Rightarrow \Delta op'Q \cong \Delta oQ_1p'$$

$$\Rightarrow \qquad \frac{\overline{p'Q}}{\overline{p'Q_1}} = \frac{\overline{op'}}{\overline{oQ_1}} = \frac{\overline{oQ}}{a} = \frac{r}{a}$$

so we can construct the *Green's* fuc. $\frac{1}{2\pi}\ln r$

$$G\ (r', \theta'; r, \theta) = \frac{1}{2\pi}\ln\left(\frac{r}{a}\frac{\overline{pQ_1}}{pQ}\right) \quad \text{for which when, } p \to p'$$

[illegible]

$$\Rightarrow \frac{r}{a}\frac{\overline{p'Q_1}}{p'Q} = 1$$

$$\Rightarrow G(a, \theta'; r, \theta) = 0$$

$$therefore \quad G(r', \theta'; r, \theta) = \frac{1}{2\pi}\ln\frac{\frac{r}{a}\left|\underset{\sim}{r}' - \frac{a^2}{r^2}\underset{\sim}{r}\right|}{\left|\underset{\sim}{r}' - \underset{\sim}{r}\right|}$$

$$\boxed{\begin{aligned}&\frac{r^2}{a^2}\left(r'^2 + \left[\left(\frac{a}{r}\right)^2\right]^2 r^2 - 2r'\frac{a^2}{r^2}\cdot r\cos(\theta' - \theta)\right)\\ &= a^2 + \frac{r^2r'^2}{a^2} - 2rr'\cos(\theta' - \theta)\end{aligned}}$$

$$= \frac{1}{2\pi}\frac{1}{2}\ln\left[\frac{a^2 + \frac{r^2r'^2}{a^2} - 2rr'\cos(\theta' - \theta)}{r'^2 + r^2 - 2rr'\cos(\theta' - \theta)}\right]$$

$$\Rightarrow \frac{\partial G}{\partial n}\Big|_{r'=a} = \frac{\partial G}{\partial r'}\Big|_{r'=a} = \frac{1}{2\pi}\frac{a^2 - r^2}{a[a^2 2ar\cos(\theta' \theta) + r^2]}$$

Hence the solution is

$$u(r, \theta) = \frac{a^2 - r^2}{2\pi}\int_0^{2\pi}\frac{f(\theta')d\theta'}{[a^2 - 2ar\cos(\theta' - \theta) + r^2]}$$

In particular

$$u(o, \theta) = \frac{a^2}{2\pi}\frac{1}{a^2}\int_0^{2\pi} f(\theta')d\theta'$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\theta')d\theta' \qquad \text{– (mean value theorem)}$$

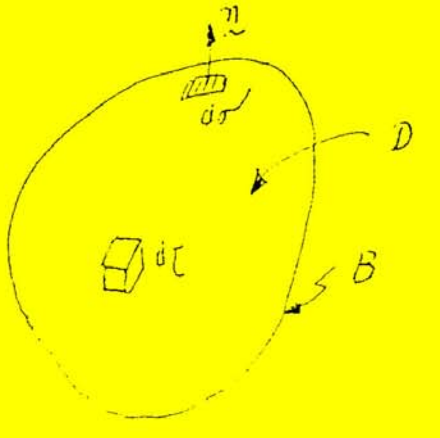
(B) 3-dim Green's fuc. ($A' \to A$)

[illegible]

(A) 3-dimensional

$$\boldsymbol{D}^2 u = \rho(\gamma) \quad in \quad D$$

$$\alpha u_n + \beta u_n = f \quad on \quad B$$



From Green's theorem

$$\int_D(\phi \boldsymbol{D}^2\psi - \psi \boldsymbol{D}^2\phi)d\tau = \oint_B\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)d\sigma$$

If we choose $\psi = G, \phi = u$

with $\boldsymbol{D}^2 G = \delta(\underset{\sim}{x}' - \underset{\sim}{x}) \quad in \quad D$

and $\boldsymbol{D}^2 u = \rho(\underset{\sim}{x}) \quad in \quad D$

$$\Rightarrow \qquad u(x) = \int_D \rho(x')G(x',x)d\tau' + \oint_B$$

For Dirichlet prob. $\beta = 0, u = f_1$ on B we demand that $G(x,x') = 0$ for x' on B then the sol. is

[illegible] 3-D [illegible]

$$u(x) = \int_D \rho(x')G(x',x)d\tau' + \oint_B f_1 \frac{\partial G}{\partial n'} d\sigma'$$

For Neumann prob. $\quad \alpha = 0, \quad \frac{\partial u}{\partial n} = f_2 \quad on \quad B$

The obvious b.c on G seems to be $\frac{\partial G}{\partial n'} = 0$

But from $\int_B \frac{\partial G}{\partial n'} d\sigma' = 0 + \int_D \nabla'^2 G d\tau$

$$= -\frac{1}{4\pi}\int_\tau \nabla^2 \frac{1}{(x-x')} d\tau$$

$$= -\frac{1}{4\pi}(-4\pi) = 1$$

$\Rightarrow$ the simplest allowable b.c on G is

$\frac{\partial G}{\partial n'} = \frac{1}{S} \quad for \quad x' \quad on \quad B \qquad$ (S : area of B surface)

Then the solution is

$$u(x) = \int_D \rho(x')G(x',x)d\tau' - \oint_B f_2 G d\sigma' + < u >_s$$

where $< u >_s = \oint_B u \frac{\partial G}{\partial n'} d\sigma' = \frac{1}{s}\oint u d\sigma'$ is the average value over the whole surface.

For infinite space $\quad s \to \infty \quad therefore \quad < u >_s \to 0$

(A') 3-d. Free Space Green's Function

$$\nabla^2 F(\underset{\sim}{x}', \underset{\sim}{x}) = \delta(\underset{\sim}{x}' - \underset{\sim}{x})$$

$$F(\underset{\sim}{x}', \underset{\sim}{x}) = F(\underset{\sim}{x}' - \underset{\sim}{x})$$

Let's translate x, compute $F(\underset{\sim}{x}', \underset{\sim}{0})$

$$i.e. \quad \nabla^2 F(\underset{\sim}{x}', 0) = \delta(\underset{\sim}{x}')$$

By 3-d. F.T.

$$\tilde{F}(k,0) = \int\int\int F(x',0) e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}'} d^3x'$$

$$\Rightarrow -k^2\tilde{F} = 1$$

$$therefore \quad \tilde{F}(k,0) = -\frac{1}{k^2}$$

Inverse T. $\Rightarrow$

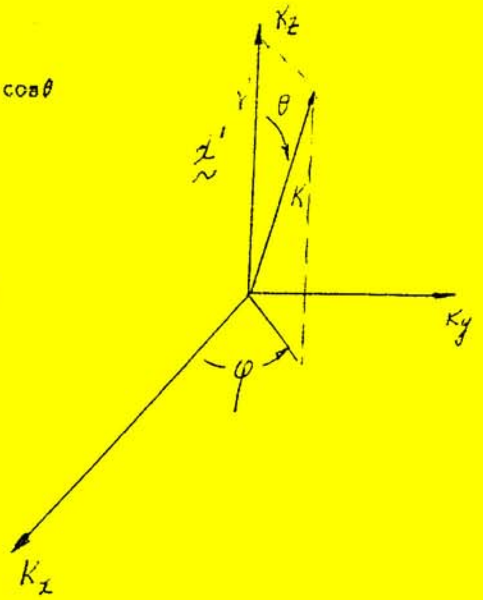
$$F(x',0) = -\frac{1}{(2\pi)^3}\int\int\int \frac{1}{k^2} e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}'} d^3k$$

$$where \quad k = |\underset{\sim}{k}|$$

choose spherical coordinates in k-space with k_z axis along $\underset{\sim}{x}'$

then $\quad d^3k = k^2 \sin\theta d\psi d\theta dk$

$$\underset{\sim}{k}\cdot\underset{\sim}{x}' = kr\cos\theta$$



therefore $\quad F = \frac{1}{(2\pi)^3}\int_0^{2\pi} d\psi \int_0^\infty dk \int_0^\pi d\theta \sin\theta e^{-ikr\cos\theta}$

$$= -\frac{1}{(2\pi)^3}\int_0^{2\pi} d\psi \int_0^\infty \left[\frac{e^{-ikr\cos\theta}}{ikr}\right]_0^\pi dK$$

$$= -\frac{1}{(2\pi)^2}\int_0^\infty \frac{dk}{ikr}[e^{ikr} - e^{-ikr}]$$

$$= -\frac{1}{2\pi^2}\int_0^\infty \frac{\sin kr}{kr} = -\frac{1}{2\pi^2 r}\int_0^\infty \frac{dv \sin v}{v}$$

$$= -\frac{1}{4\pi r}$$

therefore $\quad F(\underset{\sim}{x}', 0) = -\frac{1}{4\pi}\frac{1}{|\underset{\sim}{x}'|}$

$$\Rightarrow \quad F(\underset{\sim}{x}', \underset{\sim}{x}) = -\frac{1}{4\pi}\frac{1}{|\underset{\sim}{x}' - \underset{\sim}{x}|}$$

Ex.: Dirichlet prob. for a sphere

$$\nabla^2 u = 0 \qquad in \quad |\underset{\sim}{r}| < a$$

$$u(a, \theta, \varphi) = f(\theta, \varphi)$$

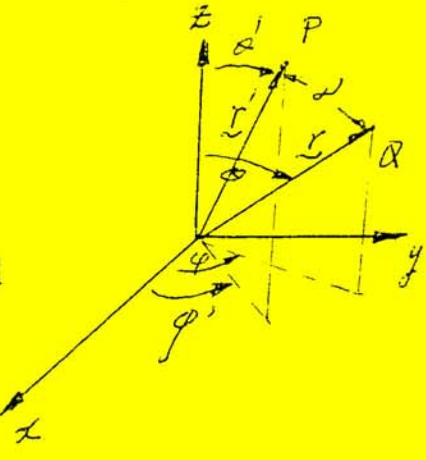
By the method of image.

The Green's fuc.

$$G(x', x) = -\frac{1}{4\pi}\left[\frac{1}{|r' - r|} - \frac{a}{r|r' - \frac{a^2}{r^2}r|}\right]$$

$$= -\frac{1}{4\pi}\left[\frac{1}{(r^2 + r' - 2rr'\cos\gamma)^{\frac{1}{2}}} - \frac{1}{(\frac{rr'^2}{a^2} + a^2 - 2rr'\cos\gamma)^{\frac{1}{2}}}\right]$$

$$\Rightarrow \frac{\partial G}{\partial n'}\Big|_{r'=a} = \frac{\partial G}{\partial r'}\Big|_{r'=a} = \frac{1}{4\pi}\frac{(r^2 - a^2)}{a(r^2 + a^2 - 2ar\cos\gamma)^{\frac{3}{2}}}$$



Hence the sol.

$$u(r, \theta, \varphi) = \frac{1}{4\pi}\int f(\theta', \varphi')\frac{a(r^2 - a^2)}{(r^2 + a^2 - 2ar\cos\gamma)^{\frac{3}{2}}} d\Omega'$$

where $d\Omega'$ is the element of solid angle on the surface, and

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$$

(D) Green's Function Expanded by Spherical Harmonics

If we require $\quad \nabla^2 G(\underset{\sim}{x}', \underset{\sim}{x}) = \delta(\underset{\sim}{x}' - \underset{\sim}{x})$

with b.c's $G(\underset{\sim}{x}', \underset{\sim}{x}) = 0$ *for* $\underset{\sim}{x}'$ *on* B $(r' = a, b)^+$

From $\quad \delta(\underset{\sim}{x}' - \underset{\sim}{x}) = \frac{1}{r'^2}\delta(r' - r)\delta(\varphi' - \varphi)\delta(\cos\theta' - \cos\theta)$

for $\quad 1 = \int \delta(\underset{\sim}{x}' - \underset{\sim}{x}) r'^2 dr' d(\cos\theta) d\varphi$

By using the completeness relation

$$\delta(\underset{\sim}{x}' - \underset{\sim}{x}) = \frac{1}{r'^2}\delta(\underset{\sim}{r}' - \underset{\sim}{r}) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}{}^*(\theta', \varphi') Y_{lm}(\theta, \varphi)$$

Expand

$$G(\underset{\sim}{x}', \underset{\sim}{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm}(\theta, \varphi) g_l(r', r) Y_{lm}{}^*(\theta', \varphi')$$

By the substitution of δ *and* G, we have

$$A_{lm}(\theta, \varphi) = Y_{lm}(\theta, \varphi)$$

and $\frac{1}{r'^2}\frac{d}{dr'}(r'^2 \frac{dg_l}{dr'}) - \frac{l(l+1)}{r'^2} g_l = -\frac{1}{r'^2}\delta(\underset{\sim}{r}' - \underset{\sim}{r})$

therefore

$$g_l(r', r) = \begin{cases} Ar'^l + Br'^{-(l+1)} & for \quad r' < r \\ Cr'^l + Dr'^{-(l+1)} & for \quad r' > r \end{cases}$$

where A,B,C,D, are fuc. of r and to be determined by b.c's. Suppose that boundary surfaces are concentric spheres for $r' = a$ *and* $r' = b$ respectively.

$$i.e. \quad g_l(r', r) = 0 \quad for \quad r' = a, b.$$

The associated homo. prob. has the trivial sol. $g(r', r) = 0$ consider the two linearly indep. sols.

$$g_{1l}(r') = r'^l - \frac{a^{2l+1}}{r'^{\,l+1}}$$

$$g_{2l}(r') = \frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}$$

which satisfy $g_1(a) = g_2(b) = 0$

The Wranskian of $g_1, g_2 \Rightarrow$

$$\begin{aligned} W(g_{1l}, g_{2l}) &= g_{1l}g_{2l}' - g_{2l}g_{1l}' \\ &= -(2l+1)\frac{1}{r'^2}[1 - (\tfrac{a}{b})^{2l+1}] \end{aligned}$$

and $p(r') = r'^2$, *let* $pw = -(2l+1)[1 - (\frac{a}{b})^{2l+1}] = Q$ Hence

$$G(x', x) = \begin{cases} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{(r'^l - \frac{a^{2l+1}}{r'^{l+1}})(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}})}{Q} Y_{lm}{}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) \\ for \quad a < r' < r \\ \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{(r^l - \frac{a^{2l+1}}{r^{l+1}})(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}})}{G} Y_{lm}{}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) \end{cases}$$

$$\Big(\qquad for \quad r < r' < b$$

or

$$G(x', x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{(r_<{}^{l} - \frac{a^{2l+1}}{r_<{}^{l+1}})(\frac{1}{r_>{}^{l+1}} - \frac{r_>{}^{l}}{b^{2l+1}})}{Q} \quad Y_{lm} * (\theta', \varphi') Y_{lm}(\theta, \varphi)$$

where $r_< = \quad min \quad (r', r)$

$r_> = \quad max \quad (r', r)$

For the special cases

(i) Infinite space $\quad a \to 0, b \to \infty$

$$G(x', x) = -\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<{}^{l}}{r_>{}^{l+1}} Y_{lm}{}^{*}(\theta', \varphi') Y_{lm}(\theta, \varphi)$$

(ii) Exterior prob. with a spherical boundary $\quad r = a, b \to \infty$

$$G(x', x) = -\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{(2l+1)} [\frac{r_<{}^{l}}{r_>{}^{l+1}} - \frac{1}{a}(\frac{a^2}{rr'})^{l+1}] Y_{lm}{}^{*} Y_{lm}$$

(iii) Interior prob. with a spherical boundary $\quad r = b, a \to 0$

$$G(x', x) = -\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{(2l+1)} [\frac{r_<{}^{l}}{r_>{}^{l+1}} - \frac{1}{b}(\frac{rr'}{b^2})^{l+1}] Y_{lm}{}^{*} Y_{lm}$$

Ex.:

Potential inside a sphere of radius b

$$D^2 u = 0$$

$$u|_{r=b} = f(\theta, \varphi)$$

Sol.:

$$u(x) = \oint_B u \frac{\partial G}{\partial n'} d\sigma'$$

$$\text{now} \quad \frac{\partial G}{\partial n'}|_{r'=b} = \frac{\partial G}{\partial r'}|_{r'=b} = \left[-\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{(2l+1)} \frac{\partial}{\partial r'} [\frac{r^l}{r'^{l+1}} - \frac{1}{b}(\frac{rr'}{b^2})^{l}] Y_{lm}{}^{*} Y_{lm} \right]_{r'=b}$$

$$= \frac{1}{b^2} \sum_{l,m} (\frac{r}{b})^{l} Y_{lm}{}^{*} Y_{lm}$$

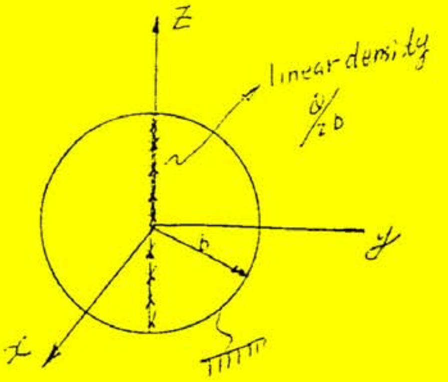
$$therefore \quad u(x) = \oint_B \frac{1}{b^2} \sum_{l,m} (\frac{r}{b})^{l} Y_{lm}{}^{*}(\theta', \varphi') Y_{lm}(\theta, \varphi) f(\theta', \varphi') b^2 d\Omega'$$

$$= \sum_{l,m} [\oint f(\theta', \varphi') Y_{lm}{}^{*}(\theta', \varphi') d\Omega'] (\frac{r}{b})^{l} Y_{lm}(\theta, \varphi)$$

Ex.:

A grounded sphere with a uniform line charge of total charge Q on the z-axis.

$$\nabla^2\phi = -4\pi\rho'$$



$$\phi|_{r=b} = 0$$

where $\rho'(x') = \frac{Q}{2b}\frac{1}{2\pi r'^2}[\delta(\cos' - 1) + \delta(\cos' + 1)]$

The factor $2\pi r^2$ in the denominator assurs that the charge density has a const. linear density $\frac{Q}{2b}$

Sol.:

$$\phi(x) = \int_0 \rho(x') G(x', x) d^3x$$

but $\rho(x') = -4\pi\rho' = -\frac{Q}{b}\frac{1}{r'^2}[\delta(\cos' - 1) + \delta(\cos' + 1)]$

therefore $\phi(x) = \int_0^b dr' \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' r'^2 \sin\theta' (-\frac{Q}{b}) \frac{1}{r'^2}[\delta(\cos\theta' - 1) + \delta(\cos\theta' + 1)]$

$$\cdot \sum_{lm} \frac{1}{2l+1}[r_<^l(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}})] Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi)$$

now $\int_0^{2\pi} e^{im\varphi'} d\varphi' = 2\pi\delta(m')$

and $Y_{lo} = Y_{lo}^* = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$

therefore $\phi(x) = \frac{Q}{b} 2\pi \frac{1}{4\pi} \sum_{l=0}^{\infty} \Big\{ \int_0^{\pi} d(\cos\theta')[\delta(\cos\theta' - 1) + \delta(\cos\theta' + 1)] P_l(\cos\theta')$

$$P_l(\cos\theta) \int_a^b r_<^l(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}) dr' \Big\}$$

$$= \frac{Q}{2b} \sum_{l=0}^{\infty} [P_l(1) + P_l(-1)] P_l(\cos\theta) \int_0^b r_<^l(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}) dr'$$

The last integral

$$I = (\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}) \int_0^r r'^l dr' + r^l \int_r^l (\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}) dr'$$

$$= \frac{2l+1}{l(l+1)}[1 - (\frac{r}{b})^l]$$

For l=0, I is indeterminate

apply L'Hospital rule

$$I = \lim_{l\to 0} \frac{\frac{d}{dl}[1 - (\frac{r}{b})^l]}{\frac{d}{dl}(l)} = \lim_{l\to 0} \left(-\frac{d}{dl}\left(e^{l \ln(\frac{r}{b})}\right)\right) = \ln(\frac{b}{r})$$

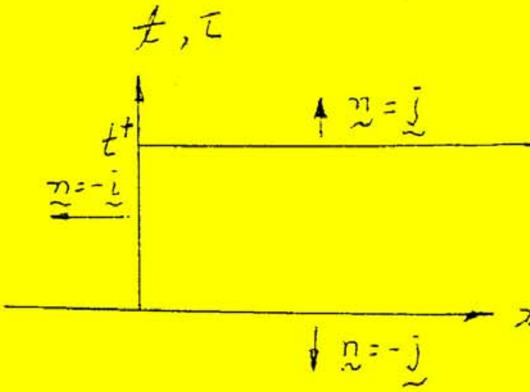
also using $P_l(-1) = (-1)^l$

$$P_l(1) = 1$$

$$\Rightarrow \phi(x) = \frac{Q}{b}\{\ln(\frac{b}{r}) + \sum_{j=1}^{\infty} \frac{4j+1}{2j(2j+1)}[1 - (\frac{r}{b})^{2j}] P_{2j}(\cos\theta)\}$$

II-VIII Green's Function Method for the Diffusion Operator

(A) 1-d: Consider a semi-infinite conducting rod.



$$\kappa u_t - u_{xx} = \Gamma(x,t)$$

I.C.:

$$u(x,0) = u_0(x)$$

B.C.:

$$u(0,t) = f(t)$$

If we choose $L^*G = -\kappa G - G_{\xi\xi}$
$= \delta(\xi - x, \tau - t)$

Then $u(x,t) = \int_R G\Gamma d\sigma - \oint_\gamma (uG_\xi - Gu_\xi)\underset{\sim}{i} + \kappa uG\underset{\sim}{j}] \cdot \underset{\sim}{n} ds$ —(1)

Let's consider a time duration t^+, the value of t^+ will be seen to be immaterial, as long as it exceeds any particular value of t at which the sol. is sought.

$$(1) \Rightarrow \quad \int_0^{t^+}\int_0^\infty G\Gamma d\sigma + \int_0^{t+}(uG_\xi - Gu_\xi)|_{\xi=0} d\tau$$
$$+ \int_0^\infty \kappa uG|_{\tau=0} d\xi - \int_0^\infty \kappa uG|_{\tau=t^+} d\xi \quad —(2)$$

Since u_ξ is not prescribed on $\xi = 0$, and u is not prescribed on $\tau = t^+$, so let's require that

$$G = 0 \quad along \quad both \quad \xi = 0 \quad and \quad \tau = t^+$$

i.e. require $\begin{cases} G(0,\tau;x,t) = 0 \\ G(\xi,\tau;x,t) = 0 \quad \text{for all } \tau > t \end{cases}$ - - - - - - (3)

Then (2) $\Rightarrow$

$u(x,t) = \int_0^t \int_0^\infty G(\xi,\tau;x,t)\Gamma(\xi,\tau) d\xi d\tau + \int_0^t f(\tau) G_\xi(0,\tau;x,t) d\tau$ $+ \int_0^\infty \kappa u_0(\xi) G(\xi, 0; x,$

- - (4)

Seek G in the form

$$G = F + g$$

where $F = \frac{H(t-\tau)e^{-\kappa(\xi-x)^2/4(t-\tau)}}{4\pi x(t-\tau)^{1/2}}$ τ > t , F ~

To satisfy(3), we can constant g by a simple image system $g = F(\xi,\tau;x,t)$, a negative unit heat pulse at (-x,t). ?

i.e. $G = F(\xi,\tau;x,t) - F(\xi,\tau;-x,t)$?

Similarly, if B.C. is $\frac{\partial u}{\partial x}(0,t) = g(t)$

we require G to satisfy

$$\frac{\partial G}{\partial \xi}(0,\tau;x,t) = 0$$

and

$$G(\xi,\tau;x,t) = 0 \quad for \quad all \quad \tau > t$$

then the sol. is given as

$$u(x,t) = \int_0^t \int_0^\infty G(\xi,\tau;x,t)\Gamma(\xi,\tau)d\xi d\tau$$
$$- \int_0^t g(\tau)G(0,\tau;x,t)d\tau + \int_0^\infty \kappa u_0(\xi)G(\xi,0;x,t)d\xi$$

Ex.: Linear heat flow in semi-infinite solid $\quad x > 0$

$$I.C : u(x,0) = f(x)$$

$$B.C : u(0,t) = \phi(t)$$

In this case

$$G(\xi,\tau;x,t) = \frac{H(t-\tau)}{2[\pi\kappa(t-\tau)]^{1/2}}[e^{-\kappa(x-\xi)^2/4(t-\tau)} - e^{-\kappa(x+\xi)^2/4(t-\tau)}]$$

$$G_\xi(\xi,\tau;x,t) = \frac{H(t-\tau)}{2[\pi\kappa(t-\tau)]^{1/2}}[\frac{-\kappa}{4(t-\tau)}]\times$$
$$[2(x-\xi)e^{-\kappa(x-\xi)^2/4(t-\tau)} - 2(x+\xi)e^{-\kappa(x+\xi)^2/4(t-\tau)}]_{\xi=0}$$
$$= \frac{H(t-\tau)}{2(\frac{\pi}{\kappa})^{1/2}}\frac{x}{(t-\tau)^{3/2}}e^{-\kappa x^2/4(t-\tau)}$$

Then (4) $\Rightarrow$

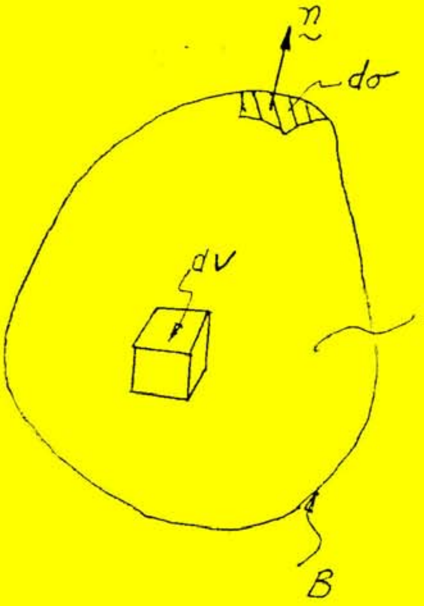
$$u(x,t) = \frac{1}{2}\sqrt{\frac{\kappa}{\pi t}}\int_0^\infty f(\xi)[e^{-\kappa(x-\xi)^2/4t} - e^{-\kappa(x+\xi)^2/4t}]d\xi$$
$$+ \frac{x}{2}\sqrt{\frac{\kappa}{\pi}}\int_0^t \phi(\tau)\frac{e^{-\kappa x^2/4(t-\tau)}}{(t-\tau)^{3/2}}d\tau$$

(B) 3-dim

$$\kappa\frac{\partial u}{\partial t} - \nabla^2 u = \Gamma(\underset{\sim}{x},t) \quad in \quad D, t > 0$$
$$I.C : \qquad u(\underset{\sim}{x},0) = u_0(\underset{\sim}{x}) \quad in \quad D + B$$

B.C : (i) *either* $u(\underset{\sim}{x}, t) = f(\underset{\sim}{x}, t)$ for $\underset{\sim}{x}$ on B

(ii) *or* $\frac{\partial u}{\partial n}(\underset{\sim}{x}, t) = g(\underset{\sim}{x}, t)$ for $\underset{\sim}{x}$ on B



(i) Require G such that

$$-\kappa \frac{\partial G}{\partial \tau} - \nabla^2 G = \delta(\underset{\sim}{\xi} - \underset{\sim}{\kappa}; \tau - t)$$

$$G(\underset{\sim}{\xi}, \tau; \underset{\sim}{x}, t) = 0 \quad for \quad \underset{\sim}{\xi} \quad on \quad B$$

and $G(\underset{\sim}{\xi}, \tau; \underset{\sim}{x}, t) = 0$ for all $\tau > t$

Then the solution is

$$u(x, t) = \int_D \kappa G(\underset{\sim}{\xi}, 0; x, t) u_0(\xi) dv$$
$$- \int_0^t \int_B f(\xi, \tau) \frac{\partial G}{\partial n'}(\xi, \tau; x, t) d\tau\, d\sigma$$
$$+ \int_0^t \int_D G(\xi, \tau; x, t) \Gamma(\xi, \tau) dv d\tau \text{——(5)}$$

(ii) Require G such that

$$-\kappa \frac{\partial G}{\partial \tau} - \nabla^2 G = \delta(\xi - x; \tau - t)$$

$$\frac{\partial G}{\partial n}(\xi, \tau; x, t) = 0 \quad for \quad \underset{\sim}{\xi} on \quad B$$

$$G(\xi, \tau; x, t) = 0 \quad for \quad all \tau > t$$

Then the 2nd term in (5) is replaced by

$$\int_0^t \int_B g(\xi, \tau) G(\xi, \tau; x, t) d\sigma d\tau$$

Ex.: Heat conduction in the rectangular parallel piped

$$0 < x < a, \quad 0 < y < b, \quad 0 < z < c$$

$$\kappa u_t - \nabla^2 u = \Gamma(x, y, z, t)$$

i.e. : $u(x, y, z, 0) = f(x, y, z)$

B.C. : Given u on the boundary.

we require

$$-\kappa G_\tau - \nabla^2 G = \delta(\tau - t)\delta(\xi - x)\delta(\eta - y)\delta(\varsigma - z)$$

and $G(\xi, \eta, \zeta, t; x, y, z, \tau) = 0$ for ξ, η, ζ on B

$$G(\xi, \eta, \varsigma, \tau; x, y, z, t) = 0 \quad \text{for all} \quad \tau > t$$

Let

$$G(\xi, \eta, \varsigma, \tau; x, y, z, t) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{lmn}(\tau, t) \sin \frac{l\pi\xi}{a} \sin \frac{l\pi x}{a} \sin \frac{m\pi\eta}{b} \sin \frac{m\pi y}{b} \sin \frac{n\pi\varsigma}{c} \sin \frac{n\pi z}{c}$$

also expand [illegible]

$$\delta(\tau - t)\delta(\xi - x)\delta(\eta - y)\delta(\varsigma - z) = \sum_{l,m,n} B_{lmn}(\tau; x, y, z, t) \sin\frac{l\pi\xi}{a} \sin\frac{m\pi\eta}{b} \sin\frac{n\pi\varsigma}{c}$$

EQ $\Rightarrow$

$$-\kappa(A_{lmn})_\tau + \pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)A_{lmn} = \delta(\tau - t)\frac{8}{abc} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c}$$

let $\quad \tau \longrightarrow \tau - t$

and the I.C. for $A_{lmn} = 0$

$$A_{lmn} = \frac{8}{abc} e^{[\kappa(\tau-t)\pi^2(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2})]} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c}$$

Hence the Green's fuc. becomes

$$G(\xi, \eta, \varsigma, \tau; x, y, z, t) = \frac{8}{abc} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\frac{l\pi\xi}{a} \sin\frac{l\pi x}{a} \sin\frac{m\pi\eta}{b} \sin\frac{m\pi y}{b}$$
$$\sin\frac{n\pi\varsigma}{c} \sin\frac{n\pi z}{c} exp\{-\kappa\pi^2(t - \tau)[\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}]\}$$

B.C.: If the face x=0 is specified g(x,y,z) and the other faces are zero.

Then the solution is

$$u(x, y, z, t) = \frac{8}{abc} \sum_{lmn} \sin\frac{lmx}{a} \sin\frac{n\pi y}{b} \sin\frac{n\pi z}{c} \int_0^t \int_0^a \int_0^b \int_0^c$$
$$\sin\frac{l\pi\xi}{a} \sin\frac{m\pi\eta}{b} \sin\frac{n\pi\varsigma}{c} \Gamma(\xi, \eta, \varsigma, \tau) e^{-\alpha_{lmn}(t-\tau)} d\tau d\xi d\eta d\varsigma$$
$$+ \frac{8\kappa}{abc} \sum_{lmn} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c} \int_0^a \int_0^b \int_0^c \sin\frac{l\pi\xi}{a} \sin\frac{m\pi\eta}{b} \sin\frac{n\pi\varsigma}{c}$$
$$f(\xi, \eta, \varsigma) e^{-\alpha_{lmn} t} d\xi d\eta d\varsigma + \frac{8}{abc} \sum_{lm\pi} \frac{l\pi}{a} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b}$$
$$\sin\frac{n\pi}{c} \int_0^t \int_0^b \int_0^c \sin\frac{m\pi\eta}{b} \sin\frac{n\pi\varsigma}{c} g(\xi, \eta, \tau) e^{-\alpha_{lmn}(t-\tau)} d\tau d\eta d\varsigma$$

$$where \quad \alpha_{lmn} = \kappa\pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{a^2}{c^2}\right)$$

$u_t - \alpha u_{xx} = 0$

$-G_\tau - \alpha G_{\xi\xi} = \delta$

解 Laplace Transform 該題

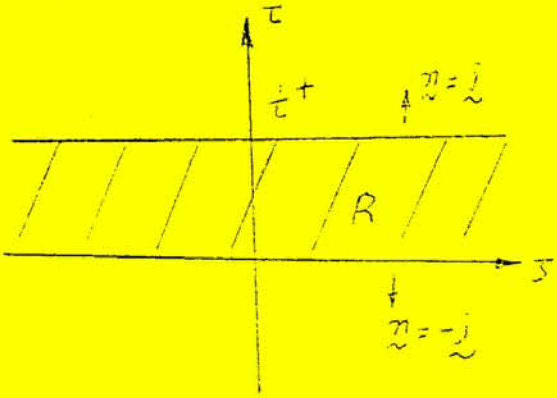
$sG + \alpha G_{\xi\xi} =$

$G(\xi, s) =$

初值解

II-9 Green's Function Method for the Wave Operator

(A). 1-dim.



$$c^2 u_{xx} - u_{tt} = \phi(x,t)$$

$$I.C.'s : u(x,0) = f(x)$$

$$u_t(x,0) = g(x)$$

If G such that:

$$L^*G = c^2 G_{\xi\xi} - G_{\tau\tau} = \delta(\xi - x; \tau - t)$$

we have

$$u(x,t) = \int_R G(\xi,\tau;x,t)\phi(\xi,\tau)d\xi d\tau$$

$$-\oint_\gamma [c^2(Gu_\xi - uG_\xi)\underset{\sim}{i} - (Gu_\tau - uG_\tau)\underset{\sim}{j}] \cdot \underset{\sim}{n} ds \quad \text{—(1)}$$

For simplicity, let $\phi = 0$

$$(1) \Rightarrow \text{u(x,t)} = -\int_{-\infty}^{\infty}(Gu_\tau - uG_\tau)|_{\tau=0} d\xi + \int_{-\infty}^{\infty}(Gu_\tau - uG_\tau)_{\tau=t^+} d\xi$$

Now we see that u, u_τ are not known at $\tau = t^+$. To remove the unwelcome terms, we therefore subject G to

$$G = G_\tau = 0 \quad or \quad \tau = t^+$$

or simply require $\quad G = 0 \quad for \quad all \tau > t^+$

$$\Rightarrow u(x,t) = \int_{-\infty}^{\infty} f(\xi)G_\tau(\xi,0;x,t) - g(\xi)G(\xi,0;x,t)]d\xi \text{ —(2)}$$

We observe that our Green fuc. is simply the "backward running" principal sol. ($\tau > t$, $G = 0$)

$$G(\xi,\tau;x,t) = -\frac{H(t-\tau)}{2c}\{H[x + c(t-\tau) - \xi] - H[x - c(t-\tau) - \xi]\}$$

and then

$$G_\tau(\xi,\tau;x,t) = \boxed{\frac{H(t-\tau)}{2}}\{\delta[x + c(t-\tau) - \xi] + \delta[x - c(t-\tau) - \xi]\}$$

$(2) \Rightarrow$

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi)\frac{H(t)}{2}\{\delta(x + ct - \xi) + \delta(x - ct - \xi)\}d\xi$$

$$+\int_{-\infty}^{\infty} g(\xi)\{\frac{H(t)}{2c}\{H(x + ct - \xi) - H(x - ct - \xi)\}d\xi$$

$$= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{-\infty}^{x+ct} g(\xi)d\xi - \frac{1}{2c}\int_{-\infty}^{x-ct} g(\xi)d\xi$$

$$= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi$$

(B.) 3-dim

$$\nabla^2 u - \frac{1}{c^2}u_{tt} = -4\pi f(x,t)$$

The Green's fuc. $G(\underset{\sim}{\xi},\tau;\underset{\sim}{x},t)$ satisfies

$$\left(\nabla^{\,2} - \frac{1}{c^2}\frac{\ ^2}{\tau^2}\right)G = -4\pi\delta(\underset{\sim}{\xi}-\underset{\sim}{x})\delta(\tau-t)$$

homo. b.c's demanded by physical consideration

Make use of Green's theorem, and integrate over time from ∞ to $t_+(>t)$

$$\Rightarrow \quad \int_0^{t+} d\tau \int_D d^3\underset{\sim}{\xi}(\phi\nabla^2\psi-\psi\nabla^{\,2}\phi)$$
$$= \int_0^{t+} d\tau \int_B d\sigma\left(\phi\frac{\partial\varphi}{\partial n'}-\psi\frac{\partial\phi}{\partial n'}\right) \quad (2)$$

Green's theorem ?

let $\psi = u, \qquad \phi = G$

LHS of (2) $\Rightarrow$

$$\int_0^{t+} d\tau \int_D d^3\xi\{G(-4\pi f + \tfrac{1}{c^2}\tfrac{\partial^2 u}{\partial t^2}) - u[-4\pi\delta(\xi-x)\delta(\tau-t)+\tfrac{1}{c^2}\tfrac{\partial^2 G}{\partial\tau^2}]\}$$
$$= \int_0^{t^+} d\tau \int_D d^3\underset{\sim}{\xi}\{4\pi u(\underset{\sim}{\xi},\tau)\delta(\underset{\sim}{\xi}-\underset{\sim}{x})\delta(\tau-t) - 4\pi f G + \tfrac{1}{c^2}(G\tfrac{\partial^2 u}{\partial\tau^2}$$
$$= 4\pi u(x,t) - 4\pi\int_0^{t+} d\tau \int_D d^3\xi f(\xi,\tau)G(\xi,\tau;x,t)$$
$$+\tfrac{1}{c^2}\int_D d^3\xi[G\tfrac{\partial u}{\partial\tau} - u\tfrac{\partial G}{\partial\tau}]_0^{t+}$$

since u, $\frac{\partial u}{\partial\tau}, G, \frac{\partial G}{\partial\tau}$ are not known at $\tau = t^+ > t$. Hence we require $G(\xi,\tau;x,t)\equiv 0$ for all $\tau > t$ and hence (2) $\Rightarrow$

$$u(x,t) = \int_0^t d\tau \int_D d^3\xi f(\xi,\tau)G(\xi,\tau;x,t)$$
$$+\tfrac{1}{4\pi c^2}\int_D d^3\xi[G\tfrac{\partial u}{\partial\tau} - u\tfrac{\partial G}{\partial\tau}]_{\tau=0}$$
$$+\tfrac{1}{4\pi}\int_0^t d\tau\int_B d\sigma(G\tfrac{\partial u}{\partial n'} - u\tfrac{\partial G}{\partial n'})$$

as to the b.c's for G, we can choose the proper one for eliminating the unwelcome terms in the 3rd integral of RHS.