Chapter III Calculus of Variations

III-I Maxima and Minima of Functions

(a) If f(x) is twice continuously differentiable on [a,b],

$$i.e. \quad f(x) \in c^2[a,b]$$

then,

nec condition for a max. (min.) of f(x) at $x_0 \in [a,b]$ is that $f'(x_0) = 0$ 必要

suff conditions for a max. (min.) of f(x) at $x_0 \in [a,b]$are that $f'(x_0) = 0$ also 充分
$f''(x_0) < 0 (> 0)$.

(b) More generally, if $f(x) \in c^2$ over closed domain D. Then nec and self conditions for a max. (min.) of f(x) at $x_0 \in D - \partial D$are that

$$\frac{\partial f}{\partial x_i}\Big|_{x=x_0} = 0 \quad i = 1, 2, ..., n \quad \text{and also}$$

that $\frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_{x=x_0}$ is a negative difinite (positive d)

(c) If $f(\underset{\sim}{X}) \in c^2$ on closed domain D

If we want to extremize $f(\underset{\sim}{x})$ subject to the constriants

$$g_i(x_1, ..., x_n) = 0, \quad i = 1, 2, ...k \quad (k < n)$$

Ex.: Find the extrema of f(x,y) subject to g(x,y) = 0

(i)1st method : by direct diff. of g

$$dg = g_x dx + g_y dy = 0$$

$$\Rightarrow dy = -\frac{g_x}{g_y} dx$$

So to extremize f

$$df = f_x dx + f_y dy = 0$$

$$\Rightarrow (f_x - f_y \frac{g_x}{g_y}) dx = 0$$

we have

$$f_x g_y - f_y g_x = 0 \quad and \quad g = 0$$

$$\Downarrow$$

to find (x_0, y_0)which is to extremize f subject to g=0

(ii)2nd method :

Let $\quad v(x, y; \lambda) = f(x, y) + \lambda g(x, y)$

$\Rightarrow$extrema of v without ant constraint

$\Leftrightarrow$extrema of f subject ot g=0 $v = f + \lambda g$

To extremize v

$$\Rightarrow \begin{Bmatrix} \frac{\partial v}{\partial x} = f_x + \lambda g_x = 0 \\ \frac{\partial v}{\partial y} = f_y + \lambda g_y = 0 \end{Bmatrix} \Rightarrow \quad f_x g_y - f_y g_x = 0$$

$$\frac{\partial v}{\partial \lambda} = g = 0$$

We obtain the same eqs. to extrimizing. Where λ is called The Lagrange multiplier.

Mehtod of Lagrange Multiplier

To find the stationary pts of f($\underset{\sim}{x}$) in $D - \partial D$ subject to constraints $g_i(\underset{\sim}{x}) = 0, \quad i = 1, 2, ..., k \quad (k < n)$

$\Leftrightarrow$ To find the stationary pts of

$v(\underset{\smile}{x}; \lambda_i) = f(\underset{\smile}{x}) + \sum_{i=1}^{k} \lambda_i g_i(x)$ without constraints .

To extremize $v(x; \lambda_i)$, we have the following eqs.

$\frac{\partial v}{\partial \underset{\smile}{x_i}} = 0 \quad i = 1, 2, ..., n$

and subject to $\frac{\partial v}{\partial \lambda_i} = 0 \quad \Rightarrow g_i = 0 \quad i = 1, 2, ..., k$

Example : consider the resl quadratic form

$Q(x_1, x_2, ..., x_n) = X^T \overleftrightarrow{A} X$

where $\underset{\sim}{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a column vector and $\overleftrightarrow{A}$ is a real and symmetric matrix .

Define a fuc.

$$f(\underset{\sim}{X}) = f(x_1, x_2, ..., x_n) = \frac{Q}{|X|^2} = \frac{X^T \overleftrightarrow{A} X}{X^T X}$$

$$= \frac{a_{mn} x_m \ x_n}{x_m x_m} \quad \text{(indicial notation).}$$

Then if $\underset{\sim}{Y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ is an eigenvector of $\overleftrightarrow{A}$ corresponding to eigenvalue λ.

$$\Rightarrow f(\underset{\sim}{Y}) = \frac{Y^T \overleftrightarrow{A} Y}{Y^T Y} = \frac{Y^T \lambda Y}{Y^T Y} = \lambda$$

and furthermore

$$\frac{\partial f(\underset{\sim}{X})}{\partial x_i}\Big|_{x_i=y_i} = \frac{(\underset{\sim}{X}^T\underset{\sim}{X})\cdot 2\overleftrightarrow{A}\underset{\sim}{X} - \underset{\sim}{X}^T\overleftrightarrow{A}\underset{\sim}{X}\cdot 2\underset{\sim}{X}}{(\underset{\sim}{X}^T\underset{\sim}{X})^2}\Big|_{\underset{\sim}{X}=\underset{\sim}{Y}}$$

$$= \frac{(\underset{\sim}{Y}^T\underset{\sim}{Y})2\lambda\underset{\sim}{Y} - \underset{\sim}{Y}^T\lambda\underset{\sim}{Y}\cdot 2\underset{\sim}{Y}}{(Y^TY)^2}$$

$$= 0$$

$\Rightarrow$ f is stationary w.r.t. small changes of $\underset{\sim}{X}$ about $\underset{\sim}{Y}$.

$\Rightarrow$suggests that the eigenvalues of $\overleftrightarrow{A}$ can be found by finding the stationary values of

Assuming $\lambda_1 \leq \lambda_2 \leq .... \leq \lambda_n$

Any vector can be expressed as linear combination of the normalized eigenvectors $\underset{\sim}{Y_i}$.

$\underset{\sim}{X} = c_i\underset{\sim}{Y_i}$

so that

$$f(\underset{\sim}{X}) = \frac{c_i\underset{\sim}{Y_i}^T\overleftrightarrow{A}c_j\underset{\sim}{Y_j}}{c_K Y_k^T c_m Y_m} = \frac{c_i\underset{\sim}{Y_i}^T c_j\lambda_j\underset{\sim}{Y_j}}{c_k c_m Y_k^T Y_m}$$

$$= \frac{\lambda_i c_i c_j \delta_{ij}}{c_k c_m \delta_{km}} = \frac{\sum_{i=1}^n \lambda_i c_i^2}{\sum_{i=1}^n c_i^2}$$

now since $\lambda_1 - f(\underset{\sim}{X}) = \frac{\sum_{i=1}^n(\lambda_1-\lambda_i)c_i^2}{\sum_{i=1}^n c_i^2}$, $\lambda_n = \frac{\sum_n \lambda c_i^2}{\sum c_i^2}$

therefore $\lambda_1 - f(\underset{\sim}{X}) = \frac{\sum_{i=1}^n(\lambda_1-\lambda_i)c_i^2}{\sum c_i^2} \leq 0$

and $\lambda_n - f(\underset{\sim}{X}) = \frac{\sum(\lambda_n-\lambda_i)c_i^2}{\sum c_i^2} \geq 0$

Hence $\lambda_1 \leq f(X) \leq \lambda_n$ global

$\Rightarrow$ one can find λ_1 by minimizing f(X)

and λ_n by maxinizing f(X)

How to find λ_2 ? We minize f($\underset{\sim}{X}$) over the subspace fo vectors orthogonal to Y_1

In this case f($\underset{\sim}{X}$) $= \frac{\sum_{i=2}^n \lambda_i c_i^2}{\sum_{i=2} c_i^2}$

$$\Rightarrow \lambda_2 \leq f(\underset{\sim}{X}) \leq \lambda_n$$

Use the method of Lagrange multiplier, we can find all eigenvalues and eigenvcetors in one step.

i.e. find the stationary value of Q($\underset{\sim}{x}$) subject to the const

$g(\underset{\sim}{X}) = \underset{\sim}{X}^T\underset{\sim}{X} - 1 = 0$

$\Rightarrow$ To extremize

$V(\vec{x}, \lambda)$

$$V(\underset{\sim}{x}) = Q(\underset{\sim}{x}) - \lambda g(\underset{\sim}{x})$$

二次式取極值 ⇒

The conditions are

$$\begin{cases} 2a_{ij}x_j - 2\lambda x_i = 0 & \Rightarrow |\overleftrightarrow{A} - \lambda \overleftrightarrow{I}| = 0 \\ & \quad\quad \Downarrow \\ and \quad x_i x_i = 1 & \text{to find eigenvalues} \end{cases}$$

Thus the eigenvectors satisfy

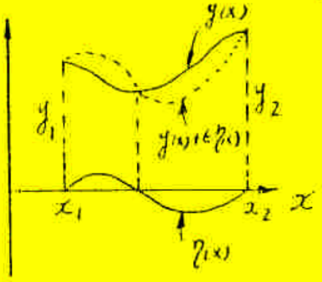
$$\overleftrightarrow{A}\underset{\sim}{Y_\alpha} = \lambda_\alpha Y_\alpha$$

$$and \qquad (\underset{\sim}{Y_\alpha}, \underset{\sim}{Y_\alpha}) = 1$$

III-2 Maxima and Minima of Functionals

We wish to find fucs depending on the entire range of one or more fucs rather than on a number of discrete vars. In other words, the domain of a functional is a set of space of admissible fucs rather than a region of coordinate space.

The simplest prob. in calculus of variations



Determine $y(x) \in c^2[x_1, x_2]$ such that the fucl.

$$I = \int_{x_1}^{x_2} F(x, y, y')dx \quad \text{as an extrema}$$

where $F \in c^2$ over its entire domain, subject to $y(x_1) = y_1, y(x_2) = y_2$ at the end pts.
Assume a fuc. $y(x) \in c^2[x_1, x_2] \quad and \quad y(x_1) = y_1, y(x_2) = y_2$, which makes I take on its minimum, then we have [illegible]

$$I(\epsilon) = \int_{x_1}^{x_2} F(x, y + \epsilon\eta, y' + \epsilon\eta')dx \geq I(0) = \int_{x_1}^{x_2} F(x, y, y')dx$$

where ϵ is arb. small parameter and η is an arb. fuc. in the class of admissible fuc.
$\eta(x) \in c^2[x_1, x_2] \quad and \quad \eta(x_1) = \eta(x_2) = 0.$ [illegible]

$\Rightarrow$ from $\quad I'(0) = 0$

$$\lim_{\epsilon\to 0} \frac{I(\epsilon) - I(0)}{\epsilon} = 0$$

i.e. $\quad \lim_{\epsilon\to 0} \int_{x_1}^{x_2} \left[\frac{F(x, y+\epsilon\eta, y'+\epsilon\eta') - F(x, y, y')}{\epsilon}\right] dx = 0$

$$\lim_{\epsilon\to 0} \int_{x_1}^{x_2} \frac{1}{\epsilon}[F_y(x, y+\epsilon\eta, y'+\epsilon\eta')\underline{\epsilon\eta} + F_{y'}(x, y+\epsilon\eta, y'+\epsilon\eta')\epsilon\eta'$$

$$+\text{terms involving higher powers in } \epsilon]dx = 0$$

$\frac{d}{d\epsilon}(y+\epsilon\eta)' = \eta$

$$\Rightarrow \int_{x_1}^{x^2} [F_y(x, y, y')\eta + F_{y'}(x, y, y')\eta']dx = 0$$

On integrating by parts of the 2nd term

$F_y \quad \Delta F$

$\Rightarrow$

$F_y - \frac{d}{dx}F_{y'}$

$$[F_{y'}(x, y, y')\eta]_{x_1}^{x_2} - \int_{x_1}^{x_2} [\frac{d}{dx}F_{y'}(x, y, y') \underline{- F_y(x, y, y')]}\eta dx = 0 \quad ------(1)$$

$since \quad \eta(x_1) = \eta(x_2) = 0$

$and \quad since \quad \eta(x) \quad is \quad arb.$

$\Rightarrow$

$\frac{d}{dx}[F_{y'}(x, y, y')] - F_y(x, y, y') = 0 \quad ------(2)$ Euler's Equation

Natural B.C's:

If the unknown fuc. y(x) is not pressigned at one or both of the end pts. x_1, x_2, for all admissible variation $\epsilon\eta(x), \eta(x_1), \eta(x_2)$ may be arbi. so if Euler eq. still hold, we must have

$$\left[\frac{\partial F}{\partial y'}\right]_{x_1} = 0 \quad or/and \quad \left[\frac{\partial F}{\partial y'}\right]_{x=x_2} = 0$$

The above requirements are called national b.c's.

Transition Conditions:

For specified $y(x_1) = y_1$, or/and $y(x_2) = y_2$, or/and natural b.c's, if $\frac{\partial F}{\partial y}$ or/and $\frac{d}{dx}(\frac{\partial F}{\partial y'})$ are discontinuous at some pt $x_0, x < x_0 < x_2$

Then eq.(1) is replaced by

$$\int_{x_1}^{x_0}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta\, dx + \int_{x_{0+}}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta\, dx$$

$$\underbrace{+\left[\frac{\partial F}{\partial y'}\eta(x)\right]_{x_1}^{x_{0-}} + \left[\frac{\partial F}{\partial y'}\eta(x)\right]_{x_{0+}}^{x_2}}_{\Downarrow} = 0$$

$$-\left(\frac{\partial F}{\partial y'}\right)_{x_1}\eta(x_1) + \left(\frac{\partial F}{\partial y'}\right)_{x_2}\eta(x_2) + \left[\left(\frac{\partial F}{\partial y'}\right)_{x_{0-}} - \left(\frac{\partial F}{\partial y'}\right)_{x_{0+}}\right]\eta(x_0)$$

If we require the fuc. y(x) to be continuous at $x = x_0$ so is the admissible fuc. $y + \epsilon\eta$

$$i.e. \quad \eta(x_{x_{0+}}) = \eta(x_{0-}) = \eta(x_0)$$

Then for extremizing the fucl I, we require

a. The Euler eq. holds in each internal $(x_1, x_0) and (x_0, x_2)$

b. Given $y(x_1) = y_1 \quad or/and \quad y(x_2) = y_2, or/and$ natural b.c's.

c. The national transition condition holds

$$y(x_{0-}) = y(x_{0+})$$

$$\left(\frac{\partial F}{\partial y'}\right)_{x_0} = \left(\frac{\partial F}{\partial y'}\right)_{x_{0+}}$$

Example:Brachistochrone with fixed end points .

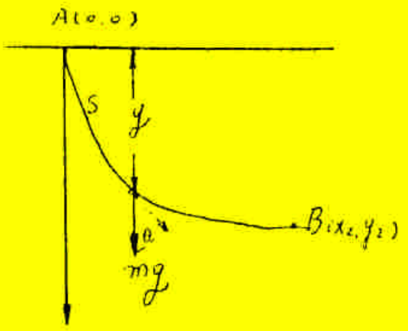
Finding the path joining two given points A and B along which a paritcle starting from rest at A should travel (under gravity and assuming no friction) in order that the time taken should be a minimum.

Let s : are length measured from A , then the eq. of motion.

$$m\ddot{s} = mg\sin\theta = mg\frac{dy}{ds}$$

since

$$\ddot{s} = \frac{d}{dt}(\dot{s}) = \frac{d}{ds}(\dot{s})\frac{ds}{dt} = \dot{s}\frac{d}{ds}(\dot{s})$$



$\Rightarrow$

$$\frac{d}{ds}\left(\frac{1}{2}\dot{s}^2\right) = g\frac{dy}{ds}$$

sol.:

$$\frac{1}{2}\dot{s}^2 = gy + c$$

when $y = 0, \dot{s} = 0 \quad \Rightarrow c = 0$

$\Rightarrow \dot{S} = 2gy$

$\Rightarrow \frac{dS}{dt} = \sqrt{2gy}$

$\Rightarrow \sqrt{2g}\, dt = \frac{ds}{\sqrt{y}}$

Special cases:

(i). F does not contain y explicity, i.e. $F_y = 0$

$$(2) \Rightarrow \quad \frac{\partial F}{\partial y'} = \quad const. \quad ------(3)$$

(ii). F does not contain x explicity,

$$using \qquad \frac{d}{dx}(y'\frac{\partial F}{\partial y'}) = y'\frac{d}{dx}(\frac{\partial F}{\partial y'}) + y''\frac{\partial F}{\partial y'}$$

$\frac{d}{dx}F_{y'} - \frac{\partial F}{\partial y} = 0$

$$(2) \Rightarrow \qquad y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'} = \frac{d}{dx}(y'\frac{\partial F}{\partial y'})$$

$$or \qquad \frac{dF}{dx} = \frac{d}{dx}(y'\frac{\partial F}{\partial y'})$$

$\left(\frac{dF}{dx} = \frac{\partial F}{\partial x} + F_y y' + F_{y'} y'' \right)$

$$\Rightarrow \qquad F - y'\frac{\partial F}{\partial y'} = const. \quad ------(4)$$

Ex.: Show that the shortest path between two pts. is a straight line.

The differential length.

$$ds = [(dx)^2 + (dy)^2]^{1/2} = [1 + y'^2]^{1/2}dx$$

$$\boxed{L = \int_a^b (1 + y'^2)^{1/2} dx}$$

$(3) \Rightarrow$

$$\frac{\partial F}{\partial y'} = \frac{y'}{(1 + y'^2)^{1/2}} = k$$

$$y'^2 = k^2(1 + y'^2)$$

$$y'^2(1-k^2) = k^2$$

or

$$y' = \frac{k}{(1-k^2)^{1/2}} \quad = \quad const$$

$\Rightarrow$ a straight line.

therefore

$$\dot{s}^2 = 2gy$$

or

$$dt = \frac{ds}{\sqrt{2gy}} \quad ------(5)$$

since $(ds)^2 = (dx)^2 + (dy)^2$

or $(ds)^2 = (dx)^2[1+(y')^2]$

(5) $\Rightarrow$

$$\sqrt{2g}t = \int_0^{x_2} [\frac{1+(y')^2}{y}]^{1/2} dx$$

To minimize t, consider $F(y,y') = [\frac{1+(y')^2}{y}]^{1/2}$ which does not contain x explicitly.

Euler eq.$\Rightarrow$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'}) = 0$$

now

$$\begin{aligned} \frac{dF}{dx} &= \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y'' \\ &= y'\frac{d}{dx}(\frac{\partial F}{\partial y'}) + y''\frac{\partial F}{\partial y'} \\ &= \frac{d}{dx}(y'\frac{\partial F}{\partial y'}) \\ F &= y'\frac{\partial F}{\partial y'} + \quad const \end{aligned}$$

[illegible]

(4) $\Rightarrow$ $F - y'\frac{\partial F}{\partial y'} \quad = \quad const$

$\Rightarrow$ $[\frac{1+(y')^2}{y}]^{1/2} - y'\frac{y'}{[y(1+y'^2)]^{1/2}} = \frac{1}{(2k)^{1/2}}$ (a convenient constant)

$$i.e. \quad y(1+y'^2) = 2k \quad ------(6)$$

This eq. can be solved for y' and then integrated . However , it is easier to proceed as follows:

put $\quad y' = \cot\varphi$

(2) $\quad \Rightarrow \quad y = \frac{2k}{1+\cot\phi^2} = 2k\sin^2\varphi = k(1-\cos 2\varphi) \quad$ —(7)

also $\quad \frac{dx}{d\varphi} = \frac{dx}{dy}\frac{dy}{d\varphi} = \frac{2k\sin 2\varphi}{y'} = \frac{2k\sin 2\varphi}{\cot\varphi} = 2k(1-\cos 2\varphi)\frac{2k\cdot 2\sin\varphi\cos\varphi}{\frac{\cos\varphi}{\sin\varphi}} = 2k\cdot 2\sin\varphi^2$

$\Rightarrow \quad x = k(2\varphi - \sin 2\varphi) + c$

But when $y = 0, x = 0$ i.e. when $\varphi = 0, x = 0 \Rightarrow c = 0$

therefore $\quad x = k(2\varphi - \sin 2\varphi) \quad ------(8)$

Thus the required curve is represented in terms of the parameter φ by (3) and (4) , and is a cycloid. The constant k is evaluated using the condition that the point $B(x_2, y_2)$ lies on the curve.

(x_1, x_2) 間有一不連續點 c

$$\int_{x_1}^{c_-}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta\,dx + \int_{c_+}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta\,dx + \left[\frac{\partial F}{\partial y'}\eta(x)\right]_{x_1}^{c_-} + \left[\frac{\partial F}{\partial y'}\eta(x)\right]_{c_+}^{x_2} = 0$$

若求極小化(或極大化)函數在 $x=c$ 不連續,

$y(x)+\epsilon\eta(x)$ 亦應具同樣性質 $\eta(c_+) = \eta(c_-) = \eta(c)$

$$\Rightarrow \int_{x_1}^{c_-} dx + \int_{c_+}^{x_2} dx + \frac{\partial F}{\partial y'}\Big|_{x_2}\eta(x_2) - \left[\frac{\partial F}{\partial y'}\right]_{x_1}\eta(x_1) - \left\{\left[\frac{\partial F}{\partial y'}\right]_{c_+} - \left[\frac{\partial F}{\partial y'}\right]_{c_-}\right\}\eta(c) = 0$$

y 若兩端是未給定

$$\Rightarrow \begin{cases} \frac{\partial F}{\partial y'}\Big|_{x_2} = \frac{\partial F}{\partial y'}\Big|_{x_1} = 0 \\ 且 \quad y(c_+) = y(c_-), \quad \lim_{x\to c_+}\frac{\partial F}{\partial y'} = \lim_{x\to c_-}\frac{\partial F}{\partial y'} \end{cases}$$

: 自然轉接條件

(natural transition condition

$y(c_+) = y(c_-) \quad \lim_{x\to c_+}\frac{\partial F}{\partial y'} = \lim_{x\to c_-}\frac{\partial F}{\partial y'}$

III-3 The Variational Notation

Imbed y(x) in a parameter family of fuc y(x)+$\epsilon\eta\ (x)$, the variation of y is defined as $\delta y = \epsilon\eta$.

The corresponding variation of F, δF to the order in ϵ is,

since $\delta F = F(x + y + \epsilon\eta, y' + \epsilon\eta') - F(x, y, y')$

$= \frac{\partial F}{\partial y}\epsilon\eta + \frac{\partial F}{\partial y'}\epsilon\eta'$+(terms involving higher powers in ϵ)

therefore $\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$

It is easy to show that δ satisfies the laws of differentiation and commute with $\frac{d}{dx}$ and $\int dx$.

- $\delta(F_1 F_2) = \delta F_1 \cdot F_2 + F_1 \cdot \delta F_2$
- $\delta\left(\frac{F_1}{F_2}\right) = \frac{F_2 \delta F_1 - F_1 \delta F_2}{F_2^2}$
- $\delta(F_1 \pm F_2) = \delta F_1 \pm \delta F_2$
- $\delta(F)^n = nF^{n-1}\delta F$

Hence $\delta I = \delta \int_{x_1}^{x_2} F(x, y, y')dx$

$= \int_{x_1}^{x_2} \delta F(x, y, y' \quad)dx$

$= \int_{x_1}^{x_2} (\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y')dx$

$= \int_{x_1}^{x_2} [\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'})]\delta y dx + [\frac{\partial F}{\partial y'}\delta y]_{x_1}^{x_2}$

Thus a stationary fuc. for a fucl is one for which the first variation = 0

For the more general cases

(a) Several unknown fuc. with one variable

$I = \int_{x_1}^{x_2} F(x, y_1, y_2, ..., y_n; y_1', y_2', ..., y_n')dx$

find $y_i(x), i = 1, 2, ..., n$ to extremize I

From

$$\delta I = \int_{x_1}^{x_2}\{(\frac{\partial F}{\partial y_1}\delta y_1 + \frac{\partial F}{\partial y_1'}\delta y_1') + ... + (\frac{\partial F}{\partial y_n} + \frac{\partial F}{\partial y_n'}\delta y_n')\}dx$$
$$= \int_{x_1}^{x_2}\{\textstyle\sum_{i=1}^{n}[\frac{\partial F}{\partial y_i} - \frac{d}{dx}(\frac{\partial F}{\partial y_i'})]\delta y_i\}dx + \sum_{i=1}^{n}[\delta y_i \frac{\partial F}{\partial y_i'}]_{x_1}^{x_2} = 0$$

Hence the necessary condition is

$\frac{\partial F}{\partial y_i} - \frac{d}{dx}(\frac{\partial F}{\partial y_i'}) = 0 \qquad i = 1, 2, ..., n$

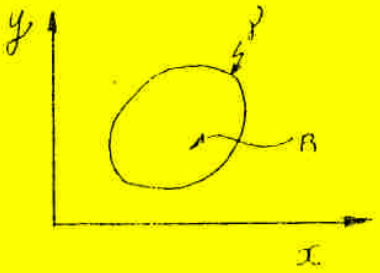
$+specified \quad b.c's: \quad y_i(x_1) = y_{i1}, y_i(x_2) = y_{i2}$

or/and national b.c's.

(b)One unknown fuc. with several vars

$I = \int\int_R F(x, y, u, u_x, u_y)dxdy$

where $u(x, y)\epsilon c^2 \quad on \quad R - \gamma$



and $u(x,y)$ prescribed , or unspecified on γ.

we vary $u(x,y) = u(x,y) + \delta u(x,y)$ with $\delta u = 0$ or arb. on γ.

$\delta I = \int_R (\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y) dxdy$

[illegible]

now

$\int_R [\frac{\partial F}{\partial u_x}\frac{\partial}{\partial x}(\delta u) + \frac{\partial F}{\partial u_y}\frac{\partial}{\partial y}(\delta u)] dxdy$

$= \int_R \{[\frac{\partial}{\partial x}(\frac{\partial F}{\partial u_x}\delta u) + \frac{\partial}{\partial y}(\frac{\partial F}{\partial u_y}\delta u)] - \delta u[\frac{\partial}{\partial x}(\frac{\partial F}{\partial u_x}) + \frac{\partial}{\partial y}(\frac{\partial F}{\partial u_y})]\} dxdy$

$= \oint_\gamma \delta u(\frac{\partial F}{\partial u_x}dy - \frac{\partial F}{\partial u_y}dx) - \int_R \delta u[\frac{\partial}{\partial x}(\frac{\partial F}{\partial u_x}) + \frac{\partial}{\partial y}(\frac{\partial F}{\partial u_y})] dxdy$

Hence $\delta I = \oint_\gamma \delta u(\frac{\partial F}{\partial u_x}dy - \frac{\partial F}{\partial u_y}dx)$

$+ \int_R \delta u[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}(\frac{\partial F}{\partial u_x}) - \frac{\partial}{\partial y}(\frac{\partial F}{\partial u_y})] dxdy = 0$

[illegible] 之 nature boundary condition

$\Rightarrow \oint_\gamma \delta u(\frac{\partial F}{\partial u_x}dy - \frac{\partial F}{\partial u_y}dx)$

Hence the Euler equation is

$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}(\frac{\partial F}{\partial u_x}) - \frac{\partial}{\partial y}(\frac{\partial F}{\partial u_y}) = 0$

+ b.c's when u is precribed on γ or/and natural b.c's when u is not specified.

(c) Several Unknowns with several vars

$I = \int_D (x_1, ...x_n, u_1, ..., u_k, u_{1x_1}, ..., u_{1x_n}, u_{2x_1}, ..., u_{2x_n}, ..., u_{kx_1}, ..., u_{kx_n}) dx$

By the same procedures

$\frac{\partial F}{\partial u_i} - \sum_{j=1}^n \frac{\partial}{\partial x_j}(\frac{\partial F}{\partial u_{ix_j}}) = 0$ $i = 1, 2 \cdots k$

+ b.c's where u_i are specified on ∂D or/and natural b.c's when u_i are specified on ∂D.

[illegible]

[illegible]

$F(x, y, y', y'')$ 之 Euler Eq $\Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'}) + \frac{d^2}{dx^2}(\frac{\partial F}{\partial y''}) = 0$

$F(x, y, \cdots y^n)$ $\Rightarrow \frac{\partial F}{\partial y} + \sum_{i=1}^n (-1)^i \frac{d^i}{dx^i}(\frac{\partial F}{\partial y^i}) = 0$

$\Downarrow$

$F(x_n, y_m, y_m^r)$

III-4 Contraints and Lagrangs Multiplier

(a) Find extreme value of

$$I = \int_{x_1}^{x_2} F(x, u, \nu, u_x, \nu_x)dx$$

where

$$u(x_1) = u_1, \qquad u(x_2) = u_2$$

$$\nu(x_1) = \nu_1, \qquad \nu(x_2) = \nu_2$$

and subject to the constraints

$$G(x, u, \nu) = 0 \quad - - - - - (1)$$

From

$$\delta I = \int_{x_1}^{x_2} \{[\frac{\partial F}{\partial \nu} - \frac{d}{dx}(\frac{\partial F}{\partial u_x})]\delta u + [\frac{\partial F}{\partial \nu} - \frac{d}{dx}(\frac{\partial F}{\partial \nu_x})]\partial \nu\} dx = 0 \quad - - - - - (2)$$

since there is a constraint on u, ν, (1), the variations ∂u and $\partial \nu$ can not be both assigned arbitrary, so we don't get two Euler's eqs.

From $\quad \delta G = \frac{\partial G}{\partial u}\delta u + \frac{\partial G}{\partial \nu}\delta \nu = 0$

$\Rightarrow \quad -\frac{G_\nu}{G_u}\partial \nu = \delta u$

So (1) $\Rightarrow$

$$\delta I = \int_{x_1}^{x_2} \{-\frac{G_\nu}{G_u}[\frac{\partial F}{\partial u} - \frac{d}{dx}(\frac{\partial F}{\partial u_x})] + [\frac{\partial F}{\partial \nu} - \frac{d}{dx}(\frac{\partial F}{\partial \nu_x})]\}\delta \nu dx = 0$$

$$\Rightarrow \quad \frac{\partial G}{\partial \nu}[\frac{\partial F}{\partial u} - \frac{d}{dx}(\frac{\partial F}{\partial u_x})] - \frac{\partial G}{\partial u}[\frac{\partial F}{\partial \nu} - \frac{d}{dx}(\frac{\partial F}{\partial \nu_x})] = 0$$

the above eq. together with (1) are to be solved for u,ν

(b) Simple Isoparametric Problem
[illegible]

To extremize

$$I = \int_{x_1}^{x_2} F(x, y, y')dx$$

subject to the constraint

$$J = \int_{x_1}^{x_2} G(x, y, y')dx = \textit{const}$$

and $y(x_1) = y_1, \quad y(x_2) = y_2$

Take the variation of a two-parameter family

$$y + \delta y = y + \epsilon_1\eta_1(x) + \epsilon_2\eta_2(x)$$

where $\eta_1(x)$, $\eta_2(x)$ are arb.fucs and satisfy

$$\eta_1(x_1) = \eta_2(x_1) = \eta_1(x_2) = \eta_2(x_2) = 0$$

Then
$$I(\epsilon_1, \epsilon_2) = \int_{x_1}^{x_2} F(x, y + \epsilon_1\eta_1 + \epsilon_2\eta_2, y' + \epsilon_1{\eta_1}' + \epsilon_2{\eta_2}')dx$$
$$J(\epsilon_1, \epsilon_2) = \int_{x_1}^{x_2} G(x, y + \epsilon_1\eta_1 + \epsilon_2\eta_2, y' + \epsilon_1{\eta_1}' + \epsilon_2{\eta_2}')dx$$

clearly, ϵ_1 and ϵ_2 are not indep. since J is to be maintained at constant value

$I(\epsilon_1, \epsilon_2)$ has extremum at $\epsilon_1 = \epsilon_2 = 0$subject to the constraint$J(\epsilon_1, \epsilon_2)|_{\epsilon_1=\epsilon_2=0} = C$

make use of method of Lagrange multiplier, this corresponds to $I + \lambda J$ has extremum at $\epsilon_1 = \epsilon_2 = 0$

i.e.
$$\frac{\partial}{\partial \epsilon_1}(I + \lambda J)|_{\epsilon_1=\epsilon_2=0} = 0$$
$$\frac{\partial}{\partial \epsilon_2}(I + \lambda J)|_{\epsilon_1=\epsilon_2=0} = 0$$
$\Rightarrow$

$\int_{x_1}^{x_2} \{[\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'})] + \lambda[\frac{\partial G}{\partial y} - \frac{d}{dx}(\frac{\partial G}{\partial y'})]\}\eta_i dx = 0 \quad i = 1, 2$

So the Euler equation is

$$\frac{\partial}{\partial y}(F + \lambda G) - \frac{d}{dx}[\frac{\partial}{\partial y'}(F + \lambda G)] = 0$$

This eq. is to be solved for y(x) in terms of λ and then substitute in $J = c$ to determine λ.

If the term $\frac{\partial G}{\partial y} - \frac{d}{dx}(\frac{\partial G}{\partial y'}) = 0$

then λ is arb, $\Rightarrow$ the constraint is trivial, and we can ignore λ.

For more general isoperimatric problem

To extremize $\quad I = \int_{x_1}^{x_2} F(x, y, y')dx$

where $\quad y(x_1) = y_1, \quad y(x_2) = y_2$

and subject to $\quad J_k = \int_{x_1}^{x_2} G_k(x, y, y') = C_k \qquad k = 1, 2, ..., N$

This prob. corresponds to extremize

$K = \int_{x_1}^{x_2} H(x, y, y')dx$ without constrainst

where $\quad H = F + \sum_{k=1}^{N} \lambda_k G_k$

Ex.: Variatonal formulation of Strum-Liouville equation . Suppose it is required to extremize the fucl

$$I = \int_{x_1}^{x_2}(py'^2 + qy^2)dx$$

subject to the constraint

$$J = \int_{x_1}^{x_2} r(x)y^2 dx = \quad const \quad - - - - - (2)$$

where p, q, r are given fucs of x

Using the notation

$$I - \lambda J = \int_{x_1}^{x_2} F(x, y, y')dx \quad ------(3)$$

where $F = py'^2 + qy^2 - \lambda r y^2$

$$\Rightarrow \delta(I - \lambda J) = [\delta y \frac{\partial F}{\partial y'}]_{x_1}^{x_2} + \int_{x_1}^{x_2} 2\delta y[\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'})]dx$$

$$= \underline{[2\delta y(py')]_{x_1}^{x_2}} + \int_{x_1}^{x_2} 2\delta y[\underline{y(q - \lambda r) - \frac{d}{dx}(py')}]dx$$

The boundary terms of the RHS vanishes in any one of the following cases

(a) The end pts $(x_1, y_1), (x_2, y_2)$ are fixed. i.e. $(\delta y)_{x_1} = (\delta y)_{x_2} = 0$

(b) The free end pt case i.e. $(py')_{x_1} = (py')_{x_2} = 0$

(c) a combination of (a) and (b) is satisfied. Under one set of these conditions,

I is stationary, if

$$\frac{d}{dx}(py') - qy + \lambda r y = 0$$

$\Rightarrow$ Solution of S-L system are those fucs which under sationary the integral I subject to the condition (2).

Returning to (3)

$$I - \lambda J = \int_{x_1}^{x_2} py'^2 dx + \int_{x_1}^{x_2} (q - \lambda r)y^2 dx$$

$$= [y(py')]_{x_1}^{x_2} - \int_{x_1}^{x_2} y\{\frac{d}{dx}(py') + (\lambda r - q)y\}dx$$

$$= [pyy']_{x_1}^{x_2} \quad \text{when I is sationary}$$

(i.e. S.L. eq. is satisfied)

If in addition

$(pyy')_{x_1} - (pyy')_{x_2} = 0$ (if b.c's on the ends are homogeneous)

then $I - \lambda J = 0$

i.e. $\lambda = \frac{I}{J} = \frac{\int_{x_1}^{x_2}(py'^2 + qy^2)dx}{\int_{x_1}^{x_2} ry^2 dx}$

eigen value

λ is the extreme value of I subject to J=const

(C) Variable End points

If the boundary of the region of integration is not completely specified, but is to be obtained together with the unknown fuction or fucs.

Ex: Determine y(x) such that

$$\delta I = \delta\int_{x_1}^{x_2} F(x,y,y')dx = 0 \quad ------(4)$$

where x_1 is fixed and $y(x_1) = y_1$ is given and x_2 is required to lie on y=g(x), a given fuc. so that

$$y(x_2) = g(x_2)$$

since x_2 may be varied, so (4) $\Rightarrow$

$$\underbrace{[F]_{x_2}\delta x_2}_{\frac{\partial I}{\partial x_1}\delta x_2} + \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right)dx = 0$$

or

$$[F]_{x_2}\delta x_2 + \left[\frac{\partial F}{\partial y'}\right]_{x_2}\delta y(x_2) + \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\,dx = 0 \text{ ——— }(5)$$

How to relate δx_2 and $\delta y(x_2)$?

Suppose the true y(x) is changed to $y + \delta y(x)$ and the corresponding change x_2 by δx_2.

$y(x_2) = g(x_2)$

$y(x_2+\delta x_2) + \delta y(x_2+\delta x_2) = g(x_2+\delta x_2)$

$y(x_2+\delta x_2) - y(x_2) + \delta y(x_2+\delta x_2) = g(x_2+\delta x_2) - g(x_2)$

$y'(x_2)\delta x_2 + \delta y(x_2) = g'(x_2)\delta x_2$

$\delta x_2 = \frac{\delta y(x_2)}{g'(x_2) - y'(x_2)}$

Thus $y(x_2+\delta x_2) + \delta y(x_2+\delta x_2) = g(x_2+\delta x_2)$

but $\quad y(x_2) = g(x_2)$

$\Rightarrow y(x_2+\delta x_2) - y(x_2) + \delta y(x_2+\delta x_2) = g(x_2+\delta x_2) - g(x_2)$

$\Rightarrow y'(x_2)\delta x_2 + \delta y(x_2) = g'(x_2)\delta x_2 +$ higher order terms

$\Rightarrow \delta x_2 = \frac{\delta y(x_2)}{g'(x_2) - y'(x_2)}$

$\delta x_2 = \left.\frac{\delta y}{g' - y'}\right|_{x=x_2}$

(5) $\Rightarrow$

$$\left[\frac{F}{g'-y'} - \frac{\partial F}{\partial y'}\right]_{x=x_2}\delta y(x_2) + \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\text{partial}F}{\partial y'}\right)\right]dx$$

This yields the Euler eq.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$y(x_1) = y_1$

$F + (g' - y')\frac{\partial F}{\partial y'} = 0$

subject to the b.c's $y(x_1) = y_1$

and $\left[F + (g' - y')\frac{\partial F}{\partial y'}\right] = 0 \quad at \quad x = x_2$called transversality condition.

Ex.:

Suppose that a brochistochrone extends from $y = g_1(x)$ to $y = g_2(x)$ then if its initial point is $A(x_1, y_1)$, the integral to be considered is now, say

$$t\sqrt{2g} = \int_{x_1}^{x_2}\left[\frac{1+y'^2}{y}\right]^{1/2}dx = \int_{x_1}^{x_2} F(y,y')dx$$

The Euler eq. $F - y'\frac{\partial F}{\partial y'} = \text{const} = c \quad ------(6)$

Since this eq. holds in particular at $x = x_1$ and x_2, so the transversality cond.

$$[F + (g_1{}' - y')\frac{\partial F}{\partial y'}]_{x=x_1} = 0 \quad \textit{and} \quad [F + (g_2{}' - y')\frac{\partial F}{\partial y'}]_{x=x_2=0} \qquad (7)$$

are simply $g_1{}' = c = g_2{}'(x_2)$

$\Rightarrow$ tangents to the transversals $y = g_1(x), y = g_2(x)$ at the pts where they meet the bischistochrone are parallel.

The solution of (6) is still a cycloid which satisfies the transv. cond. at end pts.

At pt. A. (7) $\Rightarrow \frac{1+y'^{2 1/2}}{(y-y_1)^{1/2}} + \frac{(g_1{}'-y'^2)y'}{[(y-y_1)(1+y'^2)]^{1/2}} = 0$

$\Rightarrow y'g_1{}' + 1 = 0$

$\Rightarrow$ the broch. and the curve $y = g_1$ are orthgonal

A similar deductrin applier at pt B.

III-5 Variational Problems

We will illustrate the variational formulation by considering the following examples.

(a) Small deflection of a rotating shaft.

$$\frac{d^2}{dx^2}(EI\frac{d^2y}{dx^2}) + P\frac{d^2y}{dx^2} - \rho\omega^2 y - f(x) = 0$$

where

p : axial compressive force

f(x) : transverse loading intensity

$$\int_0^l [(EIy'')'' + py'' - \rho\omega^2 y - f]\delta y dx = 0$$

$$\Rightarrow \Big[\{(EIy'')' + py'\}\delta y - EIy''\delta y'\Big]_0^l + \delta\int_0^l\Big[\frac{1}{2}EI(y'')^2 - \frac{1}{2}p(y')^2 - \frac{1}{2}\rho\omega^2 y^2 - fy\Big]dx = 0$$

Hence if the boundary constrain are

Irrotational

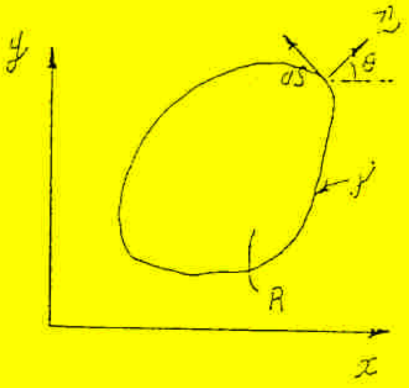
(i) Either $\delta y = 0$ (1) or $(EIy'')' + py' = 0$ (2)

and

(ii) Either $\delta y' = 0$ (3) or $EIy'' = 0$ (4)

Then the variational prob. reduces to

$$\delta\int_0^l[\frac{1}{2}EI(y'')^2 - \frac{1}{2}p(y')^2 - \frac{1}{2}\rho\omega^2 y^2 - fy]dx = 0$$



(b) Forced vibration of a membrane.

$$\frac{\partial^2 u}{\partial t^2} - c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = f(x,y,t) \quad ------(1)$$

if the forcing fuc. f is of the form

$$f(x,y,t) = P(x,y)\sin(\omega t + \alpha)$$

we may write the steady state disp u in the form

$$u = \nu(x,y)\sin(\omega t + \alpha)$$

$$(1) \Rightarrow c^2(\frac{\partial^2 \nu}{\partial x^2} + \frac{\partial^2 \nu}{\partial y^2}) + \omega^2\nu + p = 0$$

$$\int_R[c^2(\frac{\partial^2 \nu}{\partial x^2} + \frac{\partial^2 \nu}{\partial y^2}) + \omega^2\nu + p]\delta\nu dxdy = 0 \quad ------(2)$$

consider

$$c^2\int_R \nu_{xx}\delta\nu dxdy$$

$$= c^2\int_R[(\nu_x\delta\nu)_x - \nu_x\delta\nu_x]dxdy$$

$$= c^2\oint_\gamma \nu_x\delta\nu\cos\theta ds - \int_R \frac{1}{2}c^2\delta(\nu_x)^2 dxdy$$

$\cos\theta = \underset{\sim}{i}\cdot\underset{\sim}{n}$

similary $\int_R c^2\nu_{yy}\delta\nu dxdy$

$$= \oint_\gamma c^2\nu_y\delta\nu\sin\theta ds - \int_R \tfrac{1}{2}c^2\delta(\nu_y)^2dxdy$$

$(2) \Rightarrow$

$$\oint_\gamma c^2(\nu_x\cos\theta + \nu_y\sin\theta)\delta\nu ds - \int_R \tfrac{1}{2}c^2\delta[(\nu_x)^2 + (\nu_y)^2]dxdy$$
$$+ \int_R \tfrac{1}{2}\omega_2\delta(\nu^2)dxdy + \int_R P\delta\nu dxdy = 0$$
$$\Rightarrow \quad \int_\gamma c^2\frac{\partial\nu}{\partial n}\delta\nu ds - \delta\int_R[\tfrac{1}{2}c^2(\nabla\nu)^2 - \tfrac{1}{2}\omega^2\nu^2 - P\nu]dxdy = 0$$

Hence:

(i) if $\nu = f(x,y)$ is given on γ

i.e. $\delta\nu = 0 \quad on \quad \gamma$

then the variational prob $\Rightarrow$

$$\delta\int_R[\tfrac{c^2}{2}(\nabla\nu)^2 - \tfrac{1}{2}\omega^2\nu^2 - p\nu]dxdy = 0 \quad ------(3)$$

(ii) if $\frac{\partial\nu}{\partial n} = 0$ is given on γ

the variational problem is same as (3)

(iii) if $\frac{\partial\nu}{\partial n} = \psi(s)$ is given on γ

then $\Rightarrow$

$$\delta[\int_R\{\tfrac{1}{2}c^2(\nabla\nu)^2 - \tfrac{1}{2}\omega^2\nu^2 - p\nu\}dxdy - \int_\gamma c^2\psi\nu dx] = 0$$

find [illegible]

(1) D.E $\left\{\begin{matrix} \text{O.D.E} \\ \text{P.D.E} \end{matrix}\right.$ + B.C(s).

⇓

(2) δI

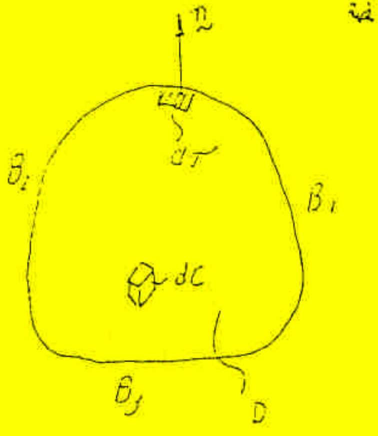
⇓

(3) $C_iS_i \Rightarrow \delta I(C_iS_i) \Rightarrow Eq$ [illegible]

(c) Steady state Heat condition

$$\nabla\cdot(\kappa\nabla T) = f(x,T) \text{ in } D$$

B.C's:



$$T = T_1 \quad on B_1$$
$$-\kappa \underset{\sim}{n}\cdot\nabla T = q_2 \quad on \quad B_2$$
$$-\kappa \underset{\sim}{n}\cdot\nabla T = h(T - T_3) \quad on \quad B_3$$

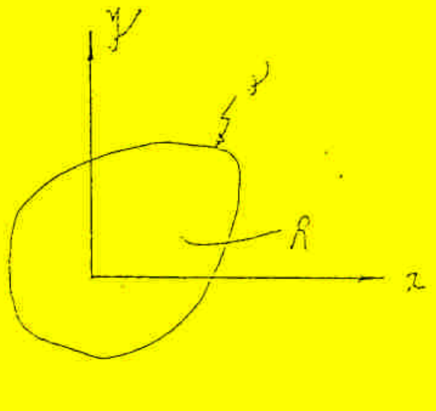
multiply the eq. by δT, and integrate over the domain D. after integrating by parts, we find the variational prob. as follow.

$$\delta[\int_D\{\tfrac{1}{2}\kappa(\nabla T)^2 + \int_{T_0}^T f(x,T')dT'\}d\tau$$
$$+ \int_{B_2} q_2Td\sigma + \tfrac{1}{2}\int_{B_3} h(T - T_3)^2d\sigma] = 0$$

with $\quad T = T_1 \quad on \quad B_1$

(d) Torsion of a primatri Bar

$$D^2\psi = -2 \quad in \quad R$$



$$\psi = 0 \quad on\gamma$$

where ψ is the Prandtl stress fuction and

$$\sigma_{\gamma x} = G_\alpha \frac{\partial\psi}{\partial y} \qquad , \qquad \sigma_{zy} = \sigma_\alpha \frac{\partial\psi}{\partial x}$$

The variational prob becomes

$$\delta\{\int_R\{(D\psi)^2 - 4\psi\}dxdy] = 0$$

with $\quad \psi = 0 \quad on\gamma$

III-6 Raleigh-Ritz Method

We have converted a b.v.p. to a variational prob. i.e. to find the solutin to extremize a fucl I subjected to the required b.c's.
When the exact solution is very difficult or impossible to obtain, the approximate solutions are discussed.

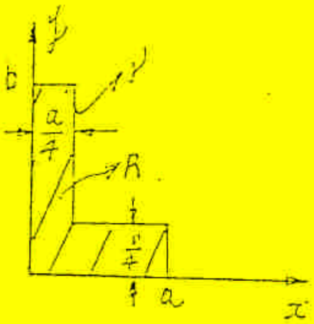
Raleigh-Ritz Method: assume the approx. solution as

$u = \sum_{i=1}^{n} C_i U_i$

where U_i are some appropriate fuc (complete set of orthogonal fucs) which satisfy the b.c's identically, then we can calculate

$I = I(c_1, ..., c_n)$, and then to choose $c_1, \sim c_n$, such that I is extreme.

$i.e. \frac{\partial I}{\partial c_1} = ... = \frac{\partial I}{\partial c_n} = 0$



Ex.: Torsion of a L-shaped Cross section shaft.

$I = \int_R [(\nabla\psi)^2 - 4\psi] dxdy \qquad \psi = 0 \quad on \quad \gamma$

Assume $\psi = A \sin(\frac{4\pi x}{a}) \sin(\frac{4\pi y}{b})$

which satisfies b.c's identically.

$$\Rightarrow \quad I(A) = \int_R [A^2 (\tfrac{4\pi}{a})^2 \cos^2(\tfrac{4\pi x}{a}) \sin^2(\tfrac{4\pi y}{b}) + A^2 (\tfrac{4\pi}{b})^2 \sin^2(\tfrac{4\pi x}{a}) \cos^2 \tfrac{4\pi y}{b}$$
$$-4A \sin(\tfrac{4\pi x}{a}) \sin(\tfrac{4\pi y}{b})] dxdy$$
$$= A^2 16\pi^2 (\tfrac{1}{a^2} + \tfrac{1}{b^2}) \cdot \tfrac{7}{64} ab + A \cdot \tfrac{ab}{\pi^2}$$

$$\frac{\partial I}{\partial A} = 0 \quad \Rightarrow \quad \tfrac{14}{4} A \pi^2 (\tfrac{1}{a^2} + \tfrac{1}{b^2}) ab + \tfrac{ab}{\pi^2} = 0$$
$$\Rightarrow A = -\frac{2}{7} \frac{a^2 b^2}{\pi^4 (a^2 + b^2)}$$

therefore $\quad \psi = -\frac{2a^2b^2}{7\pi^4(a^2+b^2)} \sin(\frac{4\pi y}{a}) \sin(\frac{4\pi y}{b})$

[illegible]題
① D.E. f $\xrightarrow{\text{find}}$ Euler Eq + B.C
② D.E $\xrightarrow{\text{find}}$ functional . special method
求近似解

Ex.2:

$$y'' + xy = -x$$

$$y(0) = y(1) = 0$$

From $\int_0^1 (y'' + xy + x) \delta y dx = 0$

$$\Rightarrow \quad [y'\delta y]_0^1 - \delta \int_0^1 [\tfrac{1}{2}(y')^2 - \tfrac{1}{2}xy^2 - xy]dx = 0$$

Hence the variation prob. becomes to extremize

$$I = \int_0^1 [\tfrac{1}{2}(y')^2 - \tfrac{1}{2}xy^2 - xy]dx$$

The appropriate assumption is

$$y = x(1-x)(c_1 + c_2x + c_3x^2 + \ldots) \quad \Leftarrow - ?$$

(i). one - term approx.

$$i.e. y_0 = c_1x(1-x) = c_1(x - x_2) \qquad y' = c_1(1-2x)$$

Then $I(c_1) = \int_0^1 [\frac{1}{2}{c_1}^2(1 - 4x + 4x^2) - \frac{x}{2}{c_1}^2(x^2 - 2x^3 + x^4) - c_1x(x - x^2)]dx$

$$= \tfrac{{c_1}^2}{2}(1 - 2 + \tfrac{4}{3}) - \tfrac{{c_1}^2}{2}(\tfrac{1}{4} - \tfrac{2}{5} + \tfrac{1}{6}) - c_1(\tfrac{1}{3} - \tfrac{1}{4})$$

$$= \tfrac{19}{120}{c_1}^2 - \tfrac{c_1}{12}$$

$$\frac{\partial I}{\partial c_1} = 0 \quad \Rightarrow \quad \tfrac{19}{60}c_1 - \tfrac{1}{12} = 0$$

$$c_1 = \tfrac{60}{19} \cdot \tfrac{1}{12} = \tfrac{5}{19} = 0.263$$

therefore $\quad y_{(1)} = 0.263x(1-x)$

(ii). two - term approx

i.e.

$$y = x(1-x)(c_1 + c_2x)$$

$$= c_1(x - x^2) + c_2(x^2 - x^3)$$

$$y' = c_1(1-2x) + c_2(2x - 3x^2)$$

Then

$$I(c_1, c_2) = \int_0^1 [\tfrac{1}{2}\{{c_1}^2(1 - 4x + 4x^2) + 2e_1c_2(2x - 7x^2 + 6x^3)$$

$$+{c_3}^2(4x^2 - 12x^3 + 9x^4)\} - \tfrac{1}{2}\{{c_1}^2(x^3 - 2x^4 + x^5) + 2c_1c_2(x^4 - 2x^5 + x^6)$$

$$+{c_2}^2(x^5 - 2x^6 + x^7)\} - \{c_1(x^2 - x^3) + c_2(x^3 - x^4)\}]dx$$

$$= \tfrac{1}{2}\{{c_1}^2(1 - 2 + \tfrac{4}{3}) + 2c_1c_2(1 - \tfrac{7}{3} + \tfrac{3}{2}) + {c_2}^2(\tfrac{4}{3} - 3 + \tfrac{9}{5})\}$$

$$-\tfrac{1}{2}\{{c_1}^2(\tfrac{1}{4} - \tfrac{2}{5} + \tfrac{1}{6}) + 2c_1c_2(\tfrac{1}{5} - \tfrac{1}{3} + \tfrac{1}{7}) + c_2^2(\tfrac{1}{6} - \tfrac{2}{7} + \tfrac{1}{8})\}$$

$$-\{c_1(\tfrac{1}{3} - \tfrac{1}{4}) + (\tfrac{1}{4} - \tfrac{1}{5})\}$$

$$= \tfrac{c_1^2}{2}(1 - 2 + \tfrac{4}{3} - \tfrac{1}{4} + \tfrac{2}{5} - \tfrac{1}{6}) + c_1c_2(1 - \tfrac{7}{3} + \tfrac{3}{2} - \tfrac{1}{5} + \tfrac{1}{3} - \tfrac{1}{7})$$

$$-\tfrac{{c_1}^2}{2}(\tfrac{4}{3} - 3 + \tfrac{9}{5} - \tfrac{1}{6} + \tfrac{2}{7} - \tfrac{9}{8}) - \tfrac{c_1}{12} - \tfrac{c_2}{20}$$

$$= \tfrac{19}{120}{c_1}^2 + \tfrac{11}{70}c_1c_2 + \tfrac{107}{1680}{c_2}^2 - \tfrac{c_1}{12} - \tfrac{c_2}{20}$$

$\frac{\partial I}{\partial c_1} = 0 \quad \Rightarrow \quad \frac{19}{60}c_1 + \frac{11}{70}c_2 = \frac{1}{12}$

i.e. $0.317c_1 + 0.157c_2 = 0.0833$

$\frac{\partial I}{\partial c_2} = 0 \quad \Rightarrow \quad \frac{11}{70}c_1 + \frac{109}{840}c_2 = \frac{1}{20}$

or $0.157\ c_1 + 0.127\ c_2 = 0.05$

$\Rightarrow c_1 = 0.177, c_2 = 0.173$

therefore $y_{(2)} = (0.177x - 0.173x^2)(1-x)$

It is noted that the deviation between the two successive approxs $y_{(1)}$ and $y_{(2)}$ is found to be smaller in magnitude than 0.009 over (0,1).

Ex.3:

$$y'' + \lambda xy = 0$$

$$y(0) = y(1) = 0$$

the exact eigenvalues is the nth solution of

$J_{1/3}(\frac{2}{3}\sqrt{\lambda}) = 0$

$\Rightarrow \quad \lambda = 18.9, 81.8$

From $\int_0^1 (y'' + \lambda xy)\delta y \, dx = 0$

$\Rightarrow$ the variational prob. $\delta I = 0$

where $I = \int_0^1 [\frac{1}{2}(y')^2 - \frac{1}{2}\lambda xy^2]dx$

or $I = \int_0^1 [(y')^2 - \lambda xy^2)dx$

(i). one - term approx.

$y = c_1 x(1-x)$

$\Rightarrow \quad I(c_1) = {c_1}^2(1 - 2 + \frac{4}{3}) - \lambda {c_1}^2(\frac{1}{4} - \frac{2}{5} + \frac{1}{6})$

$+{c_1}^2(\frac{1}{3}) - \lambda {c_1}^2(\frac{1}{60})$

$\frac{\partial I}{\partial c_1} = 0 \quad \Rightarrow \quad c_1(\frac{1}{3} - \lambda\frac{1}{60}) = 0$

$c_1 \neq 0 \Rightarrow \lambda = 20$

i.e. $\lambda_1^{(1)} = 20$

(ii). two-terms approx.

$y = x(1-x)(c_1 + c_2 x)$

$$\begin{aligned}\Rightarrow \quad I(c_1,c_2) &= {c_1}^2(1-2+\tfrac{4}{3}) + 2c_1c_2(1-\tfrac{7}{3}+\tfrac{3}{2}) + {c_2}^2(\tfrac{4}{3}-3+\tfrac{9}{5}) \\ &\quad -\lambda[{c_1}^2(\tfrac{1}{4}-\tfrac{2}{5}+\tfrac{1}{6}) + 2c_1c_2(\tfrac{1}{5}-\tfrac{1}{3}+\tfrac{1}{7}) + {c_2}^2(\tfrac{1}{6}-\tfrac{2}{7}+\tfrac{1}{8}) \\ &= \tfrac{1}{3}{c_1}^2 + \tfrac{1}{3}c_1c_2 + \tfrac{2}{15}{c_2}^2 - \lambda[\tfrac{1}{60}{c_1}^2 + \tfrac{2}{105}c_1c_2 + \tfrac{1}{168}{c_2}^2]\end{aligned}$$

$$\tfrac{\partial I}{\partial C_1} = 0 \quad \Rightarrow \quad \tfrac{2}{3}c_1 + \tfrac{1}{3}c_2 - \lambda(\tfrac{1}{30}c_1 + \tfrac{2}{105}c_2) = 0$$

$$\tfrac{\partial I}{\partial C_2} = 0 \quad \Rightarrow \quad \tfrac{1}{3}c_1 + \tfrac{4}{15}c_2 - \lambda(\tfrac{2}{105}c_1 + \tfrac{1}{84}c_2) = 0$$

or

$$(0.667 - 0.0333\lambda)c_1 + (0.333 - 0.01911\lambda)c_2 = 0$$

$$(0.333 - 0.01911\lambda)c_1 + (0.267 - 0.0119\lambda)c_2 = 0$$

$$\Rightarrow (0.667 - 0.0333\lambda)(0.267 - 0.0119\lambda) - (0.333 - 0.01911\lambda)^2 = 0$$

$$\Rightarrow {\lambda_1}^{(2)} = 19.2, \quad {\lambda_2}^{(2)} = 102$$

III-7 Estimation of Eigenvalues

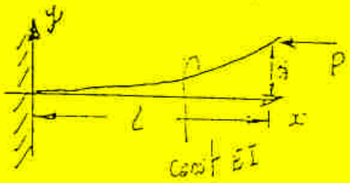
From the above example, we find

$$\lambda_1^{(1)} > \lambda_1^{(2)} > \lambda_1$$

$$\lambda_2^{(2)} > \lambda_2$$

⇒ The Rayleigh-Ritz method yield the upper bound for the eigenvalues. Why?

consider an example of buckling.



$$\text{EQ:}\quad EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = 0$$

$$\text{B.C.'s:}\quad y = \frac{dy}{dx} = 0 \quad \text{at} \quad x = 0$$

$$\left.\begin{aligned} \frac{d^2y}{dx^2} &= 0 \\ EI\frac{d^3y}{dx^3} + P\frac{dy}{dx} &= 0 \end{aligned}\right\} \quad \text{at} \quad x = l$$

From

$$\int_0^l \left(EIy'''' + Py''\right)\delta y dx = 0$$

$$\delta I = 0$$

$$\Longrightarrow \qquad I = \int_0^l \left[EI(y'')^2 - P(y')^2\right] dx \quad .$$

where P appears as the character of eigenvalue, or

$$I = \int_0^l \left[(y'')^2 - \lambda(y')^2\right] dx$$

where $\lambda = \frac{P}{EI}$

The exact critical loads is

$$P_{cr} = \frac{\pi^2 EI}{4l^2}$$

and the exact equivalent deflection cirve is

$$y = \delta\Big(1 - \cos\frac{\pi x}{2l}\Big)$$

approximation:

(i). assume the deflection is obtained from the curve of a cantilever loaded at the free end

$$\begin{aligned}
\text{i.e.}\quad y &= \frac{\delta x^2}{2l^3}(3l - x) = \frac{\delta}{2l^3}(3lx^2 - x^3)\\
\Rightarrow y' &= \frac{3\delta}{2l^3}(2lx - x^2)\\
y'' &= \frac{3\delta}{l^3}(l - x)
\end{aligned}$$

$$\begin{aligned}
I &= (\frac{3\delta}{l^3})^2 \int_0^l \Big[(l^2 - 2lx + x^2) - \lambda\cdot\frac{1}{4}(4l^2x^2 - 4lx^3 + x^4)\Big]dx\\
&= (\frac{3\delta}{l^3})^2\cdot\Big[l^3(1 - 1 + \frac{1}{3}) - \lambda\frac{l^5}{4}(\frac{4}{3} - 1 + \frac{1}{5})\Big]\\
&= (\frac{3\delta}{l^3})^2 l^3\Big[\frac{1}{3} - \frac{2}{15}\lambda l^2\Big]\\
&\frac{\partial I}{\partial\delta} = 0 \Rightarrow \lambda = \frac{5}{2}\frac{1}{l^2} = \frac{P}{EI}
\end{aligned}$$

$$\Rightarrow P = 2.5\frac{EI}{l^2}$$

(ii). assume the deflection is a parabola

i.e. $y = \frac{\delta x^2}{l^2} \quad \Rightarrow y' = \frac{2\delta}{l}x, \qquad y'' = \frac{2\delta}{l}$

$\Rightarrow$

$$\begin{aligned}
I &= (\frac{2\delta}{l})^2\int_0^l \big(1 - \lambda x^2\big)dx\\
&= (\frac{2\delta}{l})^2(l - \lambda\frac{l^3}{3})\\
\frac{\partial I}{\partial\delta} &= 0 \qquad \Rightarrow \lambda = \frac{3}{l^2}
\end{aligned}$$

$$\text{therefore}\qquad P = 3\frac{EI}{l^2}$$

(iii). assume the exact deflection curve

$$y = \delta\big(1 - \cos\tfrac{\pi x}{2l}\big)$$
$$y' = \delta(\tfrac{\pi}{2l})\sin\tfrac{\pi x}{2l}$$
$$y'' = \delta(\tfrac{\pi}{2l})^2\cos\tfrac{\pi x}{2l}$$

$$\begin{aligned} I &= \delta^2(\frac{\pi}{2l})^2\int_0^l\Big[\sin^2\frac{\pi x}{2l} - \lambda(\frac{pi}{2l})^2\cos^2\frac{\pi x}{2l}\Big]dx \\ &= \delta^2(\frac{\pi}{2l})^2\Big[(\frac{\pi}{2l})^2 - \lambda\frac{l}{2}\Big] \\ &\qquad \frac{\partial I}{\partial\delta} = 0 \qquad \Rightarrow \lambda = (\frac{\pi}{2l})^2 \end{aligned}$$

$$\Rightarrow \qquad P = \frac{\pi EI}{4l^2} = 2.4674\frac{EI}{l^2} \qquad exact!$$

The Rayleigh-Ritz method always gives values of the critical load which are larger than the true values unless the assumed curve happens to be the correct one. This follows from the fact that the true shape is the only one is equivalent . To have the column in equivalent with an incorrect shape of buckling requires that additional constraints be introduced in order to maintain that shape. The additional constraints naturally can only increase the rigidity of the column and hence the critical load becomes larger than its true value. Thus if several assumed definition curve are used, the lowest will be the most accurate.

Generally, how to find an upper bound for the smallest eigenvalue of the boundary value problem?

Consider the Helmhotz equation

$$\nabla^2\phi + \lambda\phi = 0 \qquad \text{in D}$$

$$\frac{d\phi}{dn} + \alpha\phi = 0 \qquad \text{on B}$$

From

$$\int_D\Big(\nabla^2\phi + \lambda\phi\Big)\delta\phi d\tau = 0$$

$$\Rightarrow \delta I = 0$$

assume

$$\phi_n = g = \sum_{i=1}^{n} c_i u_i$$

where u_i is any set of linear independent function in the space of addimissible function.

Denote

$$Q(f_i, f_j) = \int_D \Big(\nabla f_i \cdot \nabla f_j\Big) d\tau + \int_B \alpha f_i f_j d\sigma$$
$$N(f_i, f_j) = \int_D f_i f_j d\tau$$

where

$$Q(f) = \int_D \Big(\nabla f \cdot \nabla f\Big) d\tau + \int_B \alpha f^2 d\sigma$$
$$N(f) = \int_D f^2 d\tau$$

Then

$$I(c_i) = Q(\sum c_i u_i) - \lambda N(\sum c_i u_i)$$
$$= \sum_{i,j} a_{ij} c_i c_j - \lambda \sum_{i,j} b_{ij} c_i c_j$$

where

$$a_{ij} = Q(u_i, u_j)$$
$$b_{ij} = N(u_i, u_j)$$

From

$$\frac{\partial I}{\partial c_i} = 0 \Rightarrow |a_{ij} - \lambda b_{ij}| = 0$$

this equation gives roots, the smallest is the best upper bound for λ_1 over subspace spanned by the functions u_i $(\lambda_1^{(n)})$

i.e.

$$\lambda_1 \leq \lambda_1^{(n)} \leq \frac{Q(g)}{N(g)}$$

If we consider the K th stationary value in each of the successive approximation , we find that they form a nondecreasing sequence. This can be seen from a definition of the stationary value

$$\lambda_k^{(n)} = min \frac{Q(g)}{N(g)}$$

where g is in the subspace S spanned by $u_i, \cdots, u_n$ and $N(g, v_i) = 0$, $i = 1, 2, \cdots, k-1$, and v_i are any set of independent functions

Now

$$\lambda_k^{(n+1)} = min \frac{Q(\tilde{g})}{N(\tilde{g})}$$

where $\tilde{g}$ is in the subspace $\tilde{s}$ spanned by $u_1, \cdots, u_n, u_{n+1}$ and $N(\tilde{g}, v_i) = 0, \quad i = 1, 2, \cdots, k-1$, .
since any fuc in s can be written as a linear combination of $u_1, \cdots, u_{n+1}$ with $u_{n+1} = 0$

$\Rightarrow$ s is contained in $\tilde{s}$

$\Rightarrow \quad \lambda_k^{(n+1)} \leq \lambda_k^{(n)} \qquad k = 1, 2, \cdots, n$

$\Rightarrow \quad \lambda_k \leq \cdots \leq \lambda_k^{(n+2)} \leq \lambda_k^{(n+1)} \leq \lambda_k^{(n)}$

So we have the Raleight-Ritz array of the eigenvalues *

*	no. of admissible orthiginal func.	exact
eigenvalues	1 2 3 …… n n+1 ⋯	eigenvalues
1	$\lambda_1^{(1)} \geq \lambda_1^{(2)} \geq \lambda_1^{(3)} \cdots\cdots \geq \lambda_1^{(n)} \geq \lambda_1^{(n+1)} \geq \cdots \geq$	λ_1
2	$\lambda_2^{(2)} \geq \lambda_2^{(3)} \cdots\cdots \geq \lambda_2^{(n)} \geq \lambda_2^{(n+1)} \geq \cdots \geq$	λ_2
3	$\lambda_3^{(3)} \geq \cdots \geq \lambda_3^{(n)} \geq \lambda_3^{(n+1)} \geq \cdots \geq$	λ_3
4	$\lambda_4^{(4)} \geq \cdots \lambda_4^{(n)} \geq \lambda_4^{(n+1)} \geq \cdots \geq$	λ_4

[illegible]

III-8 Approximate Methods

(A) Method of Weighted Residuals (MWR) consider a b.v.ρ.

$L(u) \approx 0$

$\Downarrow$

$\delta I = 0 \quad (I)$

$u = \sum C_i(x_1)\, v(x_2)$

積分

$I(C_i(x), C_i'(x), \ldots)$

$\frac{\partial I}{\partial C_i} - \frac{d}{dx}\left(\frac{\partial I}{\partial C_i'}\right) = 0$

$\Downarrow$

$\mathcal{L}(C_i) = 0$

$$L[u] = 0 \quad in \quad D$$

+ homo. b.c's in B.

assume approx. solution

$$u \doteq u_n = \sum_{i=1}^{n} C_i \phi_i$$

where ϕ_i each satisfies the b.c's

The residual $R_n = L[u_n]$

In this method (MWR) , C_i are chosen such that R_n is forced to be zero in an average sense.

$i = \Delta \ni j = D$

i.e. $(\omega_j, R_n) = 0, \quad j = 1, 2, ..., n$

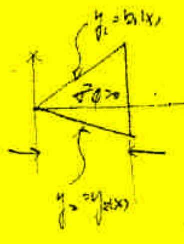
where ω_j are the weighting fucs. Corresponding to the different weighting fucs, there are subdomain method, collecation method, least square method, Galerkin method, method of moments etc.

(B) Galerkin Method

ω_j are chosen to be the trial fucs ϕ_j hence the trial fucs is chosen as members of a complete set of fucs.

A continuous fuc is zero if it is orthogonal to every number of a complete set. Thus Galerkin method fucs the residual to be zero by making it orthogonal to each of a complete set.

2-D 熱傳問題



$$\int_D R_n \phi_i d\tau = 0, \quad i = 1, 2, ..., n$$

complete set

$R_n \approx 0$

By variational principle:

Assume $\quad u_n = \sum_{i=1}^{n} C_i\phi_i \quad$ which satisfies b.c's

Then

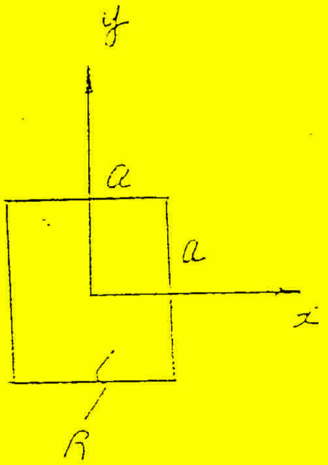
$$\int_D L[u_n]\delta u d\tau \qquad (1)$$

Galerkin idea is to choose c_i such that (1) is satisfied

(1) $\Rightarrow \int_0 R_n \sum_i \phi_i \delta c_i d\tau = 0$

or $\int_0 \sum_i R_n \phi_i \delta c_i d\tau = 0$

since δc_i are arb. $\Rightarrow$



$$\int_D R_n \phi_i d\tau = 0 \quad i = 1, 2, ..., n$$

Ex: Torsion of a square shaft

$$\nabla^2 \psi = -2$$

$$\psi = 0 \quad on \quad x = \pm a, y = \pm a$$

(i) one -term approx.

$$\psi_1 = c_1(x^2 - a^2)(y^2 - a^2)$$

$$R_i = \nabla^2 \psi_1 + 2 = 2c_1[(x-a)^2 + (y-a)^2] + 2$$

$$\phi_1 = (x^2 - a^2)(y^2 - a^2)$$

From $\int_{-a}^{a} \int_{-a}^{a} R, \phi, dxdy = 0$

$\Rightarrow c_1 = \frac{5}{8}\frac{1}{a^2}$

therefore

$\psi_1 = \frac{5}{8a^2}(x^2 - a^2)(y^2 - a^2)$

The torsional rigidity

$D_1 = 2G \int_R \psi dxdy = 0.1388G(2a)^4$

The exact value of D is

$D_a = \ .01406G(2a)^4$

the error is only -1.2%

(ii) two - term approx

$$\psi_2 = (x^2 - a^2)(y^2 - a^2)[c_1 + c_2(x^2 + y^2)]$$

$$\Downarrow$$

By symmetry → even fucs.

$\overset{\Rightarrow}{R_2} = \nabla\psi_2 + 2$

$\phi_1 = (x^2 - a^2)(y^2 - a^2)$

$\phi_2 = (x^2 - a^2)(y^2 - a^2)(x^2 + y^2)$

From $\int_R R_2\phi_1 dxdy = 0$

and $\int_R R_2\phi_2 dxdy = 0$

we obtain

$c_1 = \frac{1295}{2216}\frac{1}{a^2}, \quad c_2 = \frac{525}{4432}\frac{1}{a^2}$

therefore

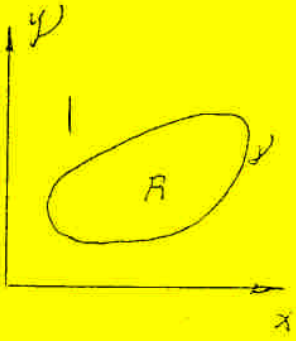
$D_2 = 2G\int_R \psi_2 dxdy = 0.1404G(2a)^4$

the error is only -0.14%

(C) Trefftz Method:

Seek the approx. sol. satisfying the eq. but not the b.c's, then minimize the error on the boundary.

For example, consider the Dinichlet integral



$$I = \int_R (\nabla u)^2 dxdy$$

$\Rightarrow \quad \delta I = 2\int_R \nabla u \cdot \nabla(\delta u)dxdy$

$= 2\int_R \nabla \cdot (\nabla u\delta u)dxdy - 2\int_R^{\nabla^2 u}\delta u dxdy$

$= 2\int_R[\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}\delta u) + \frac{\partial}{\partial y}(\frac{\partial u}{\partial y}\delta u)dxdy - 2\int_R \nabla^2 u\delta u dxdy$

$= 2\oint_\gamma \delta u(\frac{\partial u}{\partial x}dy - \frac{\partial u}{\partial y}dx) - 2\int_R \nabla^2 u\delta u dxdy$

$= 2\oint_\gamma \frac{\partial u}{\partial n}ds - 2\int_R \nabla^2 u\delta u dxdy$

From $\delta I = 0$

⇒Diridelet prob.

$$\nabla^2 u = 0 \quad in \quad G$$

$$u = f(s) \quad on \quad \gamma \quad (i.e. \quad \delta u = 0)$$

Seek sol. $u_n = \sum_{i=1}^{k} c_i v_i$

where v_i be the number of a complete set of harmonic fucs (i.e. $\nabla^2 v_i = 0$)

i.e. the assumed sol. satisfies the eq. but does not satisfy the b.c.

consider the integral

$$J = \int_R [\nabla(u - u_n)]^2 dxdy$$

and choose C_i such J is minimum

From $\frac{\partial J}{\partial C_i} = 0 \quad i = 1, 2, ..., n$

$\Rightarrow 2\int_R \nabla(u - u_n) \cdot \frac{\partial}{\partial c_i}\nabla(u - u_n)dxdy = 0$

$\Rightarrow -2\int_R \nabla(u - u_n) \cdot \nabla v_i dxdy = 0$

$\Rightarrow -2\oint_\gamma (u - u_n)\frac{\partial v_i}{\partial n}ds + 2\int_R (u - u_n)\nabla^2 v_i dxdy$

$\Rightarrow \oint_\gamma (f - \sum_{i=1}^{n} c_i v_i)\frac{\partial v_i}{\partial n}dx = 0$

$$i = 1, 2, ..., n$$

Ex.: Torsion of a square shaft

$\nabla^2\phi = 0 \quad in \quad R$

$\phi = \frac{1}{2}(x^2 + y^2) \quad on \quad \gamma$

where ϕ is the complex conjugate of the warping fuc. and is

$$\psi = \phi - \frac{1}{2}(x^2 + y^2)$$

Sol:

ϕ is harmonic even in x,y, x.y be interchanged.

Since Real and imaginarly of z^n are harmonic

	Re.	*Im.*
$n=1$	x	y
$n=2$	x^2-y^2	$2xy$
$n=4$	$x^4-6x^2y^2+y^4$	$4xy(x^2-y^2)$

take 2 -term approx.

$$\phi_2 = c_1 + c_2(x^4-6x^2y^2-y^4) \qquad \begin{cases} v_1 = 1 \\ v_2 = x^4-6x^2y^2-y^4 \end{cases}$$

From $\oint_\gamma [\frac{1}{2}(x^2+y^2)-\phi_2]\frac{\partial v_1}{\partial n}ds = 0$

instead, we require mean error $= 0$

$\Rightarrow \quad \oint_\gamma [\frac{1}{2}(x^2+y^2)-\phi_2]ds = 0$

and $\quad \oint_\gamma [\frac{1}{2}(x^2+y^2)-\phi_2] : \frac{dv_2}{dn}ds = 0$

we obtain $\quad c_1 = \frac{53}{90}a^2, c_2 = -\frac{7}{72}\frac{1}{a^2}$

$\Rightarrow \quad D = 2G\int_R \psi dxdy$

$\qquad = 0.1407G(2a)^4$