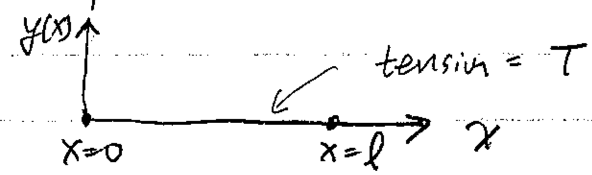
Green's function

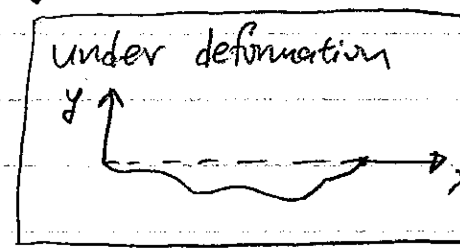
- Methods of solving linear inhomogeneous equations
  1) variation of parameters
  2) method of undetermined coefficients
  3) Green's function

- A typical inhomogeneous ordinary differential equation

$$a_0 y'' + a_1 y' + a_2 y = f(x)$$

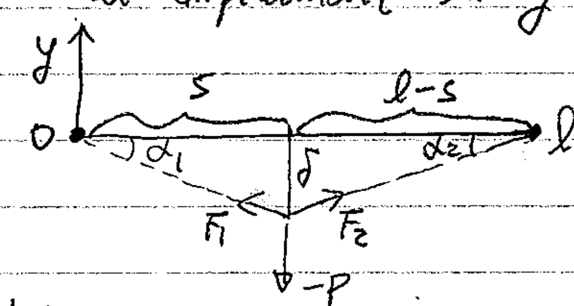
- Example: Vibration of string





B.C's: $\begin{cases} y(0)=0 \\ y(l)=0 \end{cases}$

Assumption:
- Tension in the string, prior to the load, is T.
- small displacement in y direction



Equilibrium

$$F_1 \cos\alpha_1 = F_2 \cos\alpha_2 = T$$

$$F_1 \sin\alpha_1 + F_2 \sin\alpha_2 = -P \Rightarrow \frac{F_1 \sin\alpha_1}{F_1 \cos\alpha_1} = \frac{-F_2 \sin\alpha_2 - P}{F_2 \cos\alpha_2}$$

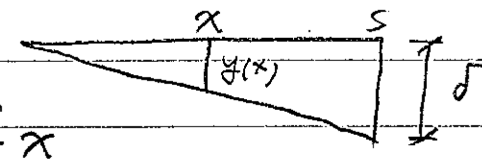
$$\Rightarrow \tan\alpha_1 = -\tan\alpha_2 - \frac{P}{T_2\cos\alpha_2} = -\tan\alpha_2 - P/T$$

$$\Rightarrow \tan\alpha_1 + \tan\alpha_2 = -P/T$$

$$\Rightarrow -\left(\frac{\delta}{s}\right) + \left(-\frac{\delta}{\ell - s}\right) = -P/T$$

$$\Rightarrow \delta = \frac{P(\ell - s)s}{T\ell}$$

Using similar triangles



$$\frac{y(x)}{x} = \frac{\delta}{s} \Rightarrow y(x) = \frac{\delta}{s}x$$

$$\Rightarrow \quad y(x,s) = \begin{cases} \dfrac{P(\ell - s)x}{T\ell}, & 0 \le x \le s \\[2ex] \dfrac{P(\ell - x)s}{T\ell}, & s \le x \le \ell \end{cases}$$

observation position

load point

If $P = 1$, then $y(x,s)$ is known as Green's function and denoted as $g(x,s)$.

Summary: $$g(x,s) = \begin{cases} \dfrac{(\ell - s)x}{T\ell} = g^-, & 0 \le x \le s \\[2ex] \dfrac{(\ell - x)s}{T\ell} = g^+, & s \le x \le \ell \end{cases}$$

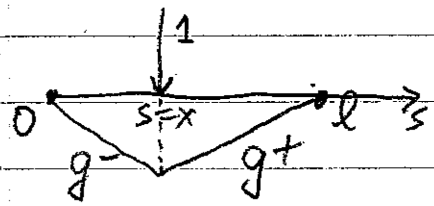
is the Green's function for equation

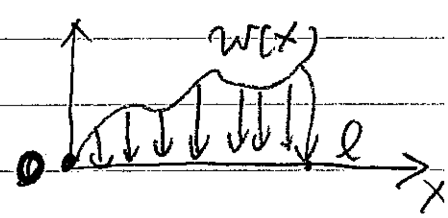
$$-Ty'' = w(x)$$

with B.C.'s

$$g(0,s) = 0$$

$$g(\ell,s) = 0$$





The Green's function solution satisfies the following interfacial continuity conditions

$$\lim_{x\to s}\left[g^+(x,s) = g^-(x,s)\right]$$

$$\lim_{x\to s}\left[\frac{\partial g^+(x,s)}{\partial x} - \frac{\partial g^-(x,s)}{\partial x}\right] = -\frac{1}{T}$$

General solution of the eqn $-Ty'' = w(x)$ is

$$y(x) = \int_0^l w(x)\, g(x,s)\, ds$$

$$= \int_0^x w(s)\, \underbrace{g^+(x,s)}_{x\geq s}\, ds + \int_x^l w(s)\, \underbrace{g^-(x,s)}_{s\geq x}\, ds$$

Note: Boundary conditions have already incorporated in the Green's function.

▨ Solving Dirichlet B.C.'s problems by Green's function Meth

① Clearly denote the two regions

② Change the non-homogeneous term (i.e. RHS) in D.(

to δ function

③ Write down the two homogeneous equation in the two region with their B.C.'s. Plus the continuity equations

$$① \quad \lim_{\underset{\sim}{x}\to\underset{\sim}{\xi}} G^{I}(\underset{\sim}{x},\underset{\sim}{\xi}) = \lim_{\underset{\sim}{x}\to\underset{\sim}{\xi}} G^{II}(\underset{\sim}{x},\underset{\sim}{\xi})$$

$$② \quad \lim_{\underset{\sim}{x}\to\underset{\sim}{\xi}}\left[\frac{dG^{II}}{d\underset{\sim}{x}} - \frac{dG^{I}}{d\underset{\sim}{x}}\right] = 1$$

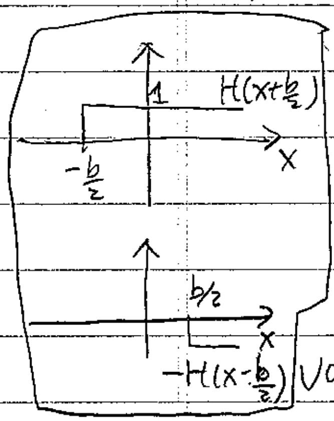
Coefficient of the highest order term

No minus sign if always keep Nonhomo. term on RHS

4. Solve the homogeneous equation in the two region (should be the same pattern, but different coefficients)
5. Use B.C.'s & continuity equations to decide unknown coefficients
6. $y(x) = \int G(\underset{\sim}{x}, \underset{\sim}{\xi}) \underbrace{f(\underset{\sim}{\xi})}_{\text{nonhomogeneous term}} d\underset{\sim}{\xi}$

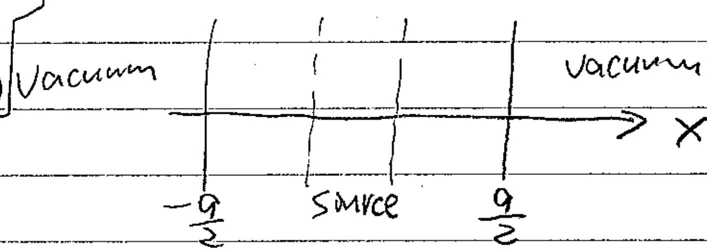
When performing the integral, be careful to use $G^{I}(\underset{\sim}{x}, \underset{\sim}{\xi})$ or $G^{II}(\underset{\sim}{x}, \underset{\sim}{\xi})$

Ex: Diffusion problem in one-dimensional planar region



$$-D\frac{d^2\phi}{dx^2} + \Sigma_0 \phi = S_0\left[H(x+\tfrac{b}{2}) - H(x-\tfrac{b}{2})\right]$$

on the interval $\left[-\frac{a}{2}, \frac{a}{2}\right]$, where $|a| > |b|$, D, Σ_0, S_0 are constant.



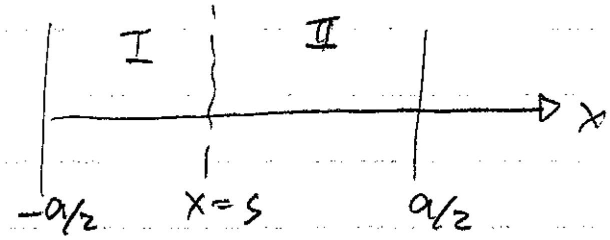
Find $\phi(x)$ everywhere inside the domain with

$$\text{B.C.'s} \quad \left.\begin{aligned} \phi(-a/2) &= 0 \\ \phi(a/2) &= 0 \end{aligned}\right\} \text{Dirichlet}$$

Sol: $-D\frac{d^2\phi}{dx^2} + \Sigma_0 \phi = \delta(x-s)$

$$\Rightarrow \frac{d^2\phi}{dx^2} - \frac{1}{L^2}\phi = -\frac{1}{D}\delta(x-s), \quad \frac{1}{L^2} = \frac{\Sigma_0}{D}$$

- Find homogeneous solution in two regions



$$\frac{d^2\phi^I}{dx^2} - \frac{1}{L^2}\phi^I = 0, \quad \phi^I(-\tfrac{a}{2}) = 0$$

$$\frac{d^2\phi^{II}}{dx^2} - \frac{1}{L^2}\phi^{II} = 0, \quad \phi^{II}(\tfrac{a}{2}) = 0$$

Continuity: $\lim_{x\to s} \phi^{II}(x>s) = \phi^I(x<x)$

$$\lim_{x\to s} \frac{d\phi^{II}}{dx} - \frac{d\phi^I}{dx} = -\frac{1}{D}$$

Region I:

$$\phi^I(x) = A\sinh\left(\tfrac{1}{L}(\tfrac{a}{2}+x)\right) + B\cosh\left(\tfrac{1}{L}(\tfrac{a}{2}+x)\right)$$

Region II:

$$\phi^{II}(x) = A'\sinh\left(\tfrac{1}{L}(\tfrac{a}{2}-x)\right) + B'\cosh\left(\tfrac{1}{L}(\tfrac{a}{2}-x)\right)$$

B.C.'s

$$\phi^I(-\tfrac{a}{2}) = 0 \Rightarrow B = 0$$

$$\phi^{II}(a/2) = 0 \Rightarrow B' = 0$$

$\Rightarrow$

$$\phi^I(x) = A\sinh\left(\tfrac{1}{L}(\tfrac{a}{2}+x)\right)$$

$$\phi^{II}(x) = A'\sinh\left(\tfrac{1}{L}(\tfrac{a}{2}-x)\right)$$

Continuity conditions ⇒

$$\lim_{x\to s} \phi^{II}(x>s) = \phi^{I}(x<s)$$

$$\Rightarrow A'\sinh\left(\tfrac{1}{L}\left(\tfrac{a}{2}-s\right)\right) = A\sinh\left(\tfrac{1}{L}\left(\tfrac{a}{2}+s\right)\right)$$

$$\Rightarrow A = A'\,\frac{\sinh\left(\frac{1}{L}\left(\frac{a}{2}-s\right)\right)}{\sinh\left(\frac{1}{L}\left(\frac{a}{2}+s\right)\right)}$$

$$\lim_{x\to s}\left(\frac{d\phi^{II}}{dx} - \frac{d\phi^{I}}{dx}\right) = -\frac{1}{D}$$

$$\Rightarrow -\frac{A'}{L}\cosh\left(\tfrac{1}{L}\left(\tfrac{a}{2}-s\right)\right) - \frac{A}{L}\cosh\left(\tfrac{1}{L}\left(\tfrac{a}{2}+s\right)\right) =$$

$$\Rightarrow A'\left[\sinh\left(\tfrac{1}{L}\left(\tfrac{a}{2}+s\right)\right)\cosh\left(\tfrac{1}{L}\left(\tfrac{a}{2}-s\right)\right) + \sinh\left(\tfrac{1}{L}\left(\tfrac{a}{2}-s\right)\right)\cosh\left(\tfrac{1}{L}\left(\tfrac{a}{2}\right.\right.\right.$$

$$= \frac{L}{D}\sinh\left(\tfrac{1}{L}\left(\tfrac{a}{2}+s\right)\right)$$

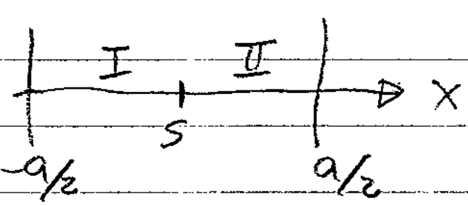
$$\Rightarrow A'\sinh\left(\frac{a}{L}\right) = \frac{L}{D}\sinh\left(\tfrac{1}{L}\left(\tfrac{a}{2}+s\right)\right)$$

$$\Rightarrow \begin{cases} A' = \dfrac{L}{D}\,\dfrac{\sinh\left(\frac{1}{L}\left(\frac{a}{2}+s\right)\right)}{\sinh\left(\frac{a}{L}\right)} \\[2ex] A = \dfrac{L}{D}\,\dfrac{\sinh\left(\frac{1}{L}\left(\frac{a}{2}-s\right)\right)}{\sinh\left(\frac{a}{L}\right)} \end{cases}$$

Green's function:

$$\phi^{I}(x) = \frac{L}{D}\,\frac{\sinh\left(\frac{1}{L}\left(\frac{a}{2}-s\right)\right)\sinh\left(\frac{1}{L}\left(\frac{a}{2}+x\right)\right)}{\sinh(a/L)},\quad x<$$

$$\phi^{II}(x) = \frac{L}{D}\,\frac{\sinh\left(\frac{1}{L}\left(\frac{a}{2}+s\right)\right)\sinh\left(\frac{1}{L}\left(\frac{a}{2}-x\right)\right)}{\sinh(a/L)},\quad x>s$$



$$g(x) = \frac{L}{D\sinh\left(\frac{a}{L}\right)}\begin{cases}\sinh\left(\frac{1}{L}\left(\frac{a}{2}-s\right)\right)\sinh\left(\frac{1}{L}\left(\frac{a}{2}+x\right)\right), & x<s \\ \sinh\left(\frac{1}{L}\left(\frac{a}{2}+s\right)\right)\sinh\left(\frac{1}{L}\left(\frac{a}{2}-x\right)\right), & x>s\end{cases}$$

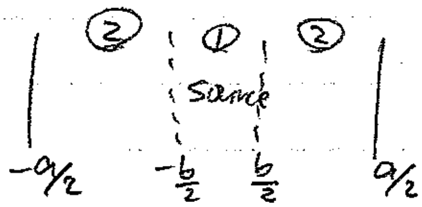
Solution of $\phi(x) \Rightarrow$

$$\phi(x) = \int_{-a/2}^{a/2} S(s)\, g(x,s)\, ds, \quad S(s) = S_0 \left(H(s+\tfrac{b}{2}) - H(s-\tfrac{b}{2}) \right)$$

$$|a| > |b|$$

☆ Find: $\phi(x)$ everywhere

- outside the source region



$$\phi(x) = \int_{-a/2}^{a/2} S_0 \left(H(s+\tfrac{b}{2}) - H(s-\tfrac{b}{2})\right) g(x,s)\, ds$$

$$= \int_{-b/2}^{b/2} S_0\, g^{\mathrm{II}}(x,s)\, ds$$

$$\Rightarrow \phi(x) = S_0 \int_{-b/2}^{b/2} \left(\frac{L}{D \sinh(\frac{a}{L})} \right) \sinh\left(\frac{1}{L}(\tfrac{a}{2}+s)\right) \sinh\left(\frac{1}{L}(\tfrac{a}{2}-s)\right.$$

$$= \frac{S_0 L}{D} \frac{\sinh(\frac{1}{L}(\frac{a}{2}-x))}{\sinh(a/L)} \underbrace{\int_{-b/2}^{b/2} \sinh\left(\tfrac{1}{L}(\tfrac{a}{2}+s)\right) ds}_{\cosh(\frac{1}{L}(\frac{a}{2}+\frac{b}{2})) - \cosh(\frac{1}{L}(\frac{a}{2}-\frac{b}{2}}$$

For $\frac{b}{2} \le |x| \le a/2$,

$$\phi(x) = \frac{S_0 L^2}{D \sinh(\frac{a}{L})} \left\{ \sinh\left(\frac{1}{L}(\tfrac{a}{2} - |x|)\right) \left[\cosh\left(\tfrac{1}{L}(\tfrac{a}{2}+\tfrac{b}{2})\right) - \cosh\left(\tfrac{1}{L}(\tfrac{a}{2}-\tfrac{b}{2})\right)\right.\right.$$

- Inside the source region

$$\phi(x) = \int_{-a/2}^{a/2} g(x,s)\, S_0 \left[H(s+\tfrac{b}{2}) - H(s-\tfrac{b}{2}) \right] ds$$

$$= S_0 \int_{-b/2}^{b/2} g(x,s)\, ds = S_0 \left[\int_{-b/2}^{x} g^{\mathrm{II}}(x,s)\, ds + \int_{x}^{b/2} g^{\mathrm{I}} \right.$$

$$= \frac{S_0 L^2}{D \sinh(a/L)} \left[\sinh\left(\tfrac{a}{L}\right) - \cosh\left(\tfrac{1}{L}(\tfrac{a}{2}-\tfrac{b}{2})\right) \left(\sinh(\tfrac{1}{L}(\tfrac{a}{2}-x)) + \sinh(\tfrac{1}{L}(\tfrac{a}{2}+x)) \right) \right.$$