# III Introduction to Integral Equation ①

Let's start from a general second order ODE

$$\mathcal{L}y + f(x)\,y(x) = -h(x)$$

Here $\mathcal{L}$ is a second order differential operator.

$$\Rightarrow \mathcal{L}y = -h(x) - f(x)\,y(x)$$

$$\Rightarrow y = -\int [h(x) + f(x)\,y(x)]\,g(x,s)\,ds$$

An integral equation

Green's function solution with the assumption that "$h(x) + f(x)\,y(x)$" is the known non-homogeneous term.

Definition: (integral equation)

$f(x) = \int_a^b K(x,t)\,\phi(t)\,dt$, Fredholm equation of the first kind

$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\,\phi(t)\,dt$, Fredholm equation of the second kind

$f(x) = \int_a^x K(x,t)\,\phi(t)\,dt$, Volterra equation of the 1st kind

$\phi(x) = f(x) + \int_a^x K(x,t)\,\phi(t)\,dt$, Volterra equation of the 2nd kind

Example: $\Sigma_s(E)\,\phi(E) = \displaystyle\int_E^{E_0} \frac{\Sigma_s(E')}{E'}\,\phi(E')\,dE' + \underbrace{S(E)}_{\text{source}}$

— Volterra eqn of 2nd kind

— Inhomogeneous

Show: $\frac{d^2y}{dx^2} + A(x)\frac{dy}{dx} + B(x)y(x) = g(x)$ To be determined

$y(0)=0$ , $y(a)=0$

can be converted to $y(x)= \int_0^a k(x,x')y(x')dx' - G(x)$ ?

Let $\Sigma_s(E)\phi(E) = F(E)$

$$\Rightarrow F(E) = \int_E^{E_0} \frac{F(E')}{E'} dE' + S(E)$$

I.C. $F(E_0) = S(E_0)$

Sol:

$$\frac{d}{dE} \Rightarrow \frac{dF(E)}{dE} = \frac{dE_0}{dE}\frac{F(E_0)}{E_0} - \frac{dE}{dE}\frac{F(E)}{E} + \int_E^{E_0}\frac{d}{dE}\left(\frac{F(E')}{E'}\right)dE' + \frac{dS(E)}{dE}$$

$$\Rightarrow \frac{dF(E)}{dE} = -\frac{F(E)}{E_0} + \frac{dS(E)}{dE}$$

Note: Leibniz's rule

$$\frac{d}{dx}\int_{f(x)}^{g(x)} h(x,t)dt = g' h(x,g(x)) - f' h(x,f(x)) + \int_{f(x)}^{g(x)}\frac{\partial h}{\partial x}dt$$

$$\Rightarrow \frac{dF(E)}{dE} + \frac{1}{E}F(E) = \frac{dS(E)}{dE}$$

Integrating factor: $e^{\ln(\frac{1}{E})} = E^{-1}$

$$\Rightarrow \int_E^{E_0} d(EF(E)) = \int_E^{E_0} E'\frac{dS(E')}{dE'}dE' + \int_E^{E_0} E' dS(E')$$

$$\Rightarrow E_0F(E_0) - EF(E) = E'S(E')\Big|_E^{E_0} - \int_E^{E_0} S(E')dE'$$

I.C. $F(E_0) = S(E_0) \Rightarrow E_0F(E_0) = E_0S(E_0)$

$$\Rightarrow EF(E) = ES(E) + \int_E^{E_0} S(E')dE'$$

$$\Rightarrow F(E) = S(E) + \frac{1}{E}\int_E^{E_0} S(E')dE'$$

$$\Rightarrow \phi(E) = \frac{S(E)}{\Sigma_s(E)} - \frac{1}{\Sigma_s(E)E}\int_E^{E_0} S(E')dE'$$

3. Numerical solution of integral equations

$$\phi(x) = \lambda \int_a^b k(x,x')\,\phi(x')\,dx' + H(x)$$

From numerical integration rules

$$I = \frac{h}{3}\left[f_0(x_0) + 4 f_1(x_1) + f_2(x_2)\right]$$ Simpson's rule

or

$$I = \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + f_N\right]$$ composite Simpson's rule

$$\phi(x) = \lambda \sum_{n=0}^{3} w_n k(x,x_n)\,\phi(x_n) + H(x)$$

$$\Rightarrow \phi(x) - \lambda \sum_{n=0}^{3} w_n k(x,x_n)\,\phi(x_n) = H(x)$$

$$\Rightarrow \begin{cases} \phi(x_0) - \lambda\left[w_0 k(x_0,x_0)\phi(x_0) + w_1 k(x_0,x_1)\phi(x_1) + w_2 k(x_0,x_2)\phi(x_2) + w_3 k(x_0,x_3)\phi(x_3)\right] = H(x_0) \\ \phi(x_1) - \lambda[\qquad\qquad] = H(x_1) \\ \phi(x_2) - \lambda[\qquad\qquad] = H(x_2) \\ \phi(x_3) - \lambda[\qquad\qquad] = H(x_3) \end{cases}$$

$$\Rightarrow A_{ij}\,\phi_j = H_i$$

$$A_{ij} = \begin{cases} -\lambda w_j k(x_i,x_j), & i \neq j \\ 1 - \lambda w_j k(x_i,x_j), & i = j \end{cases}$$

$\Rightarrow$ Formal solution $\underset{\sim}{\phi} = \underset{\approx}{A}^{-1}\underset{\sim}{H}$

Fredholm eqn: $\underset{\approx}{A}$ is a full matrix

Volterra eqn: $\underset{\approx}{A}$ is a triangular matrix

4 Neumann series techniques

$$\phi(x) = \lambda \int_a^b K(x,x')\,\phi(x')\,dx' + H(x)$$

Neumann series solution $\Big\}$ $\phi(x) = \sum_{l=0}^{\infty} \phi^l(x)$

$$\phi^0(x) = H(x)$$

$$\phi^1(x) = \lambda \int_a^b K(x,x')\,\phi^0(x')\,dx'$$

$$\vdots$$

$$\phi^n(x) = \lambda \int_a^b K(x,x')\,\phi^{n-1}(x')\,dx'$$

Convergence check

$$\frac{\sum_{l=0}^{\infty} \left( \phi^l(x) - \phi^{l-1}(x) \right)}{\sum_{l=0}^{\infty} \phi^l(x)} < \varepsilon$$

If $\int_a^b K(x,x')\,dx' < 1$, convergence is gaurenteed.

Banach Contraction Mapping Theorem
(Banach-Caccioppoli Fixed-Point Theorem)

Let $\mathcal{F}$ be a closed subset of a Banach space, and let $\underline{f}: \mathcal{F} \to \mathcal{F}$ be a contraction. then

$$\underline{x} = \underline{f}(\underline{x})$$

has a unique solution $\bar{\underline{x}}$ in $\mathcal{F}$.