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Introduction to Functional Analysis

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Preface

The present introduction to functional analysis addresses students in mathematics and physics who have some basic knowledge in analysis and linear algebra. It grew out of the lectures which have been given by the authors several times.

The book is divided into four parts and an appendix. In Part I the necessary notions and results on vector spaces, metric and topological spaces, as well as on compact topological spaces, are provided.

In Part II we present the classical fundamentals of functional analysis. After introducing Banach and Fréchet spaces we prove the Hahn–Banach theorem and apply it and the bipolar theorem to study dual and bidual spaces as well as the closed range theorem. As consequences of Baire's theorem we prove the open mapping theorem, the closed graph theorem and the principle of uniform boundedness. After introducing Hilbert spaces we deal with the spaces $L_p(X,\mu)$ and $C(X)'$ and we study Fourier transform and Sobolev spaces extensively.

Part III is devoted to the spectral theory of linear operators. Beginning with Riesz's theory of compact operators in Banach and Hilbert spaces we discuss in detail Hilbert–Schmidt and trace class operators. The construction of spectral measures for normal operators in Hilbert spaces is prepared by a chapter on Banach algebras where we also treat C^*-algebras and Gelfand's theory. After proving the spectral representation for normal operators we deduce the corresponding result for (unbounded) self-adjoint operators from it using the Cayley transform. Also, we present von Neumann's theory of self-adjoint extensions of symmetric operators.

In Part IV we introduce locally convex spaces, their duality theory and we characterize reflexive spaces. Further, we treat inductive and projective topologies, Schwartz and (LF)-spaces as well as notions related to them and we prove the closed graph theorem of de Wilde. Then we concentrate on Fréchet and (DF)-spaces, where we also include recent results on the exactness of short sequences of Fréchet spaces. Next a comprehensive presentation of the Köthe sequence spaces illustrates many notions introduced so far and provides important examples and counter-examples. After a short introduction to nuclear spaces we systematically present power series spaces. Then we prove the (DN)-(Ω)-splitting theorem which is closely related to power series spaces of infinite type and which is used to characterize the subspaces and the quotients of the space s of all rapidly decreasing sequences.

In the appendix we give a short introduction to integration theory by means

1 Linear algebra

In this chapter we shall collect together the relevant notions from linear algebra that we shall need later. We denote by $\mathbb{K}$ either of the fields $\mathbb{R}$ or $\mathbb{C}$.

A *linear space* E over $\mathbb{K}$, also called a *vector space* over $\mathbb{K}$, is a nonempty set E, in which an addition $+ : E \times E \longrightarrow E$ and a scalar multiplication $\cdot : \mathbb{K} \times E \longrightarrow E$ with the following properties are defined:

1. $(E, +)$ is an Abelian group with the zero element 0.
2. $\lambda(x+y) = \lambda x + \lambda y$ for all $\lambda \in \mathbb{K}$ and $x, y \in E$.
3. $(\lambda + \mu)x = \lambda x + \mu x$ for all $\lambda, \mu \in \mathbb{K}$, $x \in E$.
4. $(\lambda\mu)x = \lambda(\mu x)$ for all $\lambda, \mu \in \mathbb{K}$, $x \in E$.
5. $1x = x$ for all $x \in E$.

The elements of E are called *vectors*.

Linear subspaces: A nonempty subset F of a $\mathbb{K}$–vector space E is called a *linear subspace* of E if $\lambda x + \mu y \in F$ for all $x, y \in F$ and $\lambda, \mu \in \mathbb{K}$.

Since the intersection of linear subspaces of a vector space E is again a linear subspace, there exists, corresponding to each subset M of E, a smallest linear subspace, span(M), which contains M. This subspace span(M) is called the *linear hull* of M. It is then obvious that:

$$\begin{aligned} \operatorname{span}(M) &= \bigcap \{F : F \text{ is a linear subspace of } E \text{ with } F \supset M\} \\ &= \Big\{ \sum_{\text{finite}} \lambda_j m_j : \lambda_j \in \mathbb{K}, m_j \in M \Big\} . \end{aligned}$$

Linear independence: Let E be a $\mathbb{K}$–vector space. A nonempty finite subset $M = \{x_1, \ldots, x_m\}$ of E is said to be *linearly independent* if $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$ and $\sum_{j=1}^m \lambda_j x_j = 0$ imply $\lambda_1 = \ldots = \lambda_m = 0$. A nonempty subset M of E is said to be *linearly independent* if each finite subset of it is linearly independent. M is said to be *linearly dependent* if M is not linearly independent.

Basis: Let E be a $\mathbb{K}$–vector space. A subset B of E is said to be a *basis* each $x \in E$ has a unique representation $x = \sum_{b \in B} \lambda_b b$, where only finitely many $\lambda_b \neq 0$.

Dimension: The *dimension* of a $\mathbb{K}$–vector space E is defined as

$$\dim_{\mathbb{K}} E := \begin{cases} 0 & \text{if } E = \{0\} \\ n & \text{if } E \text{ has a basis of } n \text{ vectors} \\ \infty & \text{if } E \text{ contains an infinite linearly independent set.} \end{cases}$$

This is meaningful, since, in the case when a finite basis exists, all bases have the same number of elements. E is said to be *finite dimensional* if $\dim_{\mathbb{K}} E \in \mathbb{N}_0$. If $\dim_{\mathbb{K}} E = \infty$, then E is said to be *infinite dimensional.*

As we will presently show, the existence of bases for infinite dimensional spaces is a consequence of Zorn's lemma. We shall treat Zorn's lemma as an axiom; its equivalence with the Axiom of Choice and the Well-Ordering Principle are well-known (see Hermes, Chapter 30 (1967); Hewitt and Stromberg, Chapter 14 (1965)).

An *order relation* is a relation $\prec$ on a set $Z \neq \emptyset$ satisfying the following:

1. $x \prec x$ for all $x \in Z$.
2. $x \prec y$ and $y \prec z$ imply $x \prec z$.
3. $x \prec y$ and $y \prec x$ imply $x = y$.

An *ordered set* is a pair $(Z, \prec)$, where $\prec$ is an order relation on Z.

Let $(Z, \prec)$ be an ordered set. An element m in Z is said to be *maximal*, if $z \in Z$ and $m \prec z$ imply $m = z$. If A is a subset of Z, then $s \in Z$ is called an *upper bound* of A if $a \prec s$ holds for each $a \in A$. $A \subset Z$ is called a *chain* if for each pair x, y of distinct elements of A either $x \prec y$ or $y \prec x$ holds. $(Z, \prec)$ is said to be *inductively ordered*, if every chain in Z has an upper bound.

The following statement is equivalent to the Axiom of Choice.

Zorn's lemma 1.1 *Every inductively ordered set has at least one maximal element.*

Proposition 1.2 *Let $E \neq \{0\}$ be a $\mathbb{K}$–vector space and $B_0 \subset E$ be a linearly independent set. Then there exists a basis B of E with $B_0 \subset B$. In particular, E has a basis.*

Proof Let

$$Z := \{M \subset E \;:\; M \text{ is linearly independent and } B_0 \subset M\}\,.$$

Considering "$\subset$" as an order relation "$\prec$", $(Z, \prec)$ is an ordered set. As can be

that $(Z, \prec)$ has a maximal element B. If we assume that $\mathrm{span}(B) \neq E$ then there exists an element $x_0 \in E \setminus \mathrm{span}(B)$. Then $B \cup \{x_0\}$ is a linearly independent set since $\mu x_0 + \sum \lambda_b b = 0$ implies

$$\mu x_0 = -\sum \lambda_b b \;\in\; \mathrm{span}(x_0) \cap \mathrm{span}(B) = \{0\}\,.$$

Consequently $B \cup \{x_0\}$ in Z is an upper bound of B, contradicting the maximality of B. Thus $\mathrm{span}(B) = E$, i.e., B is a basis of E. □

Linear maps: Let E and F be $\mathbb{K}$–vector spaces. A *linear map* A of E into F is a map $A : E \longrightarrow F$ such that

$$A(\lambda x + \mu y) = \lambda A(x) + \mu A(y) \;\text{ for all } x, y \in E,\; \lambda, \mu \in \mathbb{K}\,.$$

Frequently, we shall also refer to linear maps as (linear) *operators*. The identity map on E is linear and will be denoted by id or I. For a linear map A its kernel (null space) and range are defined as follows:

$$\ker A := N(A) := \{x \in E \;:\; Ax = 0\}$$
$$\mathrm{im}\, A := R(A) := \{Ax \;:\; x \in E\}\,.$$

Clearly $N(A)$ and $R(A)$ are linear subspaces of E and F respectively.

A linear map $A : E \longrightarrow \mathbb{K}$ is called a *linear functional*, or also, a *linear form* on E. The set of all linear forms on E is called the *algebraic dual* E^* of E. E^* becomes a $\mathbb{K}$–vector space with the following definitions of addition and scalar multiplication:

$$y + z : x \longmapsto y(x) + z(x),\; \lambda y : x \longmapsto \lambda y(x); \quad y, z \in E^*,\; \lambda \in \mathbb{K},\; x \in E\,.$$

If $A : E \longrightarrow F$ is a linear map, then so is the map $A^t y := y \circ A$ in E^* for each $y \in F^*$. The map $A^t : F^* \longrightarrow E^*$ defined as above is linear and is called the *adjoint map* of A.

Quotients: Let E be a $\mathbb{K}$–vector space and F be a linear subspace of E. The elements x and y in E are said to be equivalent ($x \sim y$), if $x - y$ is in F and this defines an equivalence relation on E. The set E/F of all equivalence classes $x + F$ of this equivalence relation becomes a $\mathbb{K}$–vector space under the following (unambiguously) defined operations of addition and scalar multiplication:

$$(x + F) + (y + F) := (x + y) + F \quad \text{and} \quad \lambda \cdot (x + F) := \lambda x + F$$

and E/F is called the *quotient vector space* of E modulo F. The map $q : E \longrightarrow E/F$, $q(x) := x + F$ is called the *quotient map* and it is linear

Corresponding to each $\mathbb{K}$–vector space G and each linear map $T : E \longrightarrow G$ with $N(T) \supset F$ there exists exactly one linear map $\overline{T} : E/F \longrightarrow G$ satisfying $T = \overline{T} \circ q$. If $N(T) = F$, then $\overline{T}$ is injective.

The *codimension* of F in E is defined by $\operatorname{codim} F := \dim E/F$.

Direct product: Let $(E_i)_{i\in I}$ be a family of $\mathbb{K}$–vector spaces. The product (set) $\prod_{i\in I} E_i$ becomes a $\mathbb{K}$–vector space, called the *direct product* of vector spaces E_i, $i \in I$, with addition and scalar multiplication defined as follows:

$$(x_i)_{i\in I} + (y_i)_{i\in I} := (x_i + y_i)_{i\in I} \quad \text{and} \quad \lambda(x_i)_{i\in I} := (\lambda x_i)_{i\in I}.$$

For $i \in I$ define the canonical map $\pi_i : \prod_{j\in I} E_j \longrightarrow E_i$ by setting $\pi_i((x_j)_{j\in I}) := x_i$. Evidently π_i is linear. The product of the vector spaces $E_i, i \in I$, has the following property:

For every $\mathbb{K}$–vector space F and for each family $(T_i)_{i\in I}$ of linear maps $T_i : F \longrightarrow E_i$ there exists a unique linear map $T : F \longrightarrow \prod_{i\in I} E_i$ such that $\pi_i \circ T = T_i$ for all $i \in I$.

Direct sum: Let $(E_i)_{i\in I}$ be a family of $\mathbb{K}$–vector spaces. The following linear subspace of $\prod_{i\in I} E_i$ is called the *direct sum* $\bigoplus_{i\in I} E_i$ of the vector spaces $E_i, i \in I$:

$$\bigoplus_{i\in I} E_i := \left\{ (x_i)_{i\in I} \in \prod_{i\in I} E_i \ : \ x_i \neq 0 \text{ only for finitely many } i \in I \right\}.$$

For $i \in I$ the map $j_i : E_i \longrightarrow \bigoplus_{\alpha\in I} E_\alpha$, $j_i(x_i) := (\delta_{\alpha,i} x_i)_{\alpha\in I}$ is called the canonical map of E_i into the direct sum. Obviously j_i is linear. $\bigoplus_{i\in I} E_i$ has the following property:

For each $\mathbb{K}$–vector space G and each family $(T_i)_{i\in I}$ of linear maps $T_i : E_i \longrightarrow G$ there exists a unique linear map $T : \bigoplus_{i\in I} E_i \longrightarrow G$ with $T \circ j_i = T_i$ for all $i \in I$.

Convex and absolutely convex sets: A subset M of a $\mathbb{K}$–vector space E is said to be *convex* if $\lambda x + (1-\lambda)y \in M$ for all $x, y \in M$ and $\lambda \in [0,1]$.

M is said to be *absolutely convex* if $M \neq \emptyset$ and $\lambda x + \mu y \in M$ for all $x, y \in M$ and all $\lambda, \mu \in \mathbb{K}$ with $|\lambda| + |\mu| \leq 1$.

Clearly the intersection of arbitrarily many convex sets is convex; therefore, for every subset X of E there exists a smallest convex set M containing X. This set is called the *convex hull* of X and is denoted by $\operatorname{conv}(X)$. It may easily be shown that

$$\left\{ \sum^{n} \ldots \sum^{n} \ldots 1, n \in \mathbb{N} \right\}$$

Analogously, for every subset X of E the *absolutely convex hull* of X, which is denoted by ΓX, exists and

$$\Gamma X = \left\{ \sum_{j=1}^{n} \lambda_j x_j : \lambda_j \in \mathbb{K}, x_j \in X, 1 \leq j \leq n, \sum_{j=1}^{n} |\lambda_j| \leq 1, n \in \mathbb{N} \right\}.$$

For every absolutely convex set X of E, $0 \in X$ and $\lambda X = |\lambda| X$ for all $\lambda \in \mathbb{K}$. Moreover, for each $a \in E$, the set $a + X := \{a + x : x \in X\}$ is convex.

Exercises:

1. Let E, F be $\mathbb{K}$–vector spaces, B a basis for E and $(f_b)_{b\in B}$ be a family in F. Show that there exists a unique linear map $A : E \longrightarrow F$ such that $A(b) = f_b$ for all $b \in B$.
2. Let E be a $\mathbb{K}$–vector space, F be a linear subspace of E and $q : E \longrightarrow E/F$ be the quotient map. Show that:
   (a) $q^* : (E/F)^* \longrightarrow E^*, q^*(y) := y \circ q$, is linear and injective and $R(q^*) = \{y \in E^* : y|_F \equiv 0\}$.
   (b) $\rho : E^* \longrightarrow F^*, \rho(y) := y|_F$ is linear and surjective.
3. Let E be a $\mathbb{K}$–vector space and X be a subset of E. Show that:
   (a) X is convex if, and only if, for all $n \in \mathbb{N}$, for all $x_1, \ldots, x_n \in X$ and all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_+$ with $\sum_{j=1}^{n} \lambda_j = 1$, $\sum_{j=1}^{n} \lambda_j x_j$ is also in X .
   (b) X is absolutely convex if, and only if, for each $n \in \mathbb{N}$, for all $x_1, \ldots, x_n \in X$ and all $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ with $\sum_{j=1}^{n} |\lambda_j| \leq 1$, $\sum_{j=1}^{n} \lambda_j x_j$ is also in X .
4. Let E be a $\mathbb{C}$–vector space. Then E can also be considered as an $\mathbb{R}$–vector space $E_{\mathbb{R}}$ in which multiplication is restricted to real scalars. Show that:
   (a) For every $y \in E^*$, $u := \operatorname{Re} y$ is in $(E_{\mathbb{R}})^*$, and $y(x) = u(x) - i\,u(ix)$ for all $x \in E$. Further, for every absolutely convex subset A of E , $\sup_{x\in A} |y(x)| = \sup_{x\in A} |u(x)|$.
   (b) For each $u \in (E_{\mathbb{R}})^*$, $y : x \longmapsto u(x) - i\,u(ix)$, $x \in E$, is in E^*.

2

Metric and topological spaces

In order to be able to cruise through functional analysis one needs, in the general set up of metric or topological spaces, the notions of continuity and convergence learnt in earlier analysis courses. The relevant notions, definitions, and facts are briefly introduced here and it is recommended that the reader carries out the simple proofs unless they are already known.

Metric spaces: Let X be a set. A *metric* on X is a function $d : X \times X \longrightarrow \mathbb{R}_+$ (where $\mathbb{R}_+ := [0,\infty[$) with the following properties:

(M1) $d(x,y) = d(y,x)$ for all $x,y \in X$ (symmetry).
(M2) $d(x,z) \le d(x,y) + d(y,z)$ for all $x,y,z \in X$ (triangle inequality).
(M3) $d(x,y) = 0$ if, and only if, $x = y$.

A *metric space* (X,d) is a nonempty set X on which a metric d is given. In the sequel we shall speak of metric spaces X without specific mention of the metric d.

In a metric space X for $a \in X$ and $\varepsilon > 0$ the set

$$U_\varepsilon(a) := \{x \in X : d(x,a) < \varepsilon\}$$

is called the *ε–neighborhood* of the point a. A subset M of X is said to be *open* if for each $a \in M$ there exists an $\varepsilon > 0$ such that $U_\varepsilon(a) \subset M$. From (M1) and (M2) it follows that all sets $U_\varepsilon(a)$ are open and that the system $\mathcal{O}$ of all open subsets of X has the following properties:

($\mathcal{O}$1) The union of arbitrarily many open sets is open; $\emptyset$ is open.
($\mathcal{O}$2) The intersection of finitely many open sets is open; X is open.
($\mathcal{O}$3) For each pair $x,y \in X$ with $x \ne y$ there exist disjoint open sets U_x and U_y with $x \in U_x$ und $y \in U_y$.

Topological spaces: Let X be a set. A *topology* on X is a system $\mathcal{O}$ of subsets of X which has the properties ($\mathcal{O}$1)–($\mathcal{O}$3) and in this case the elements of $\mathcal{O}$ are called open sets. A *topological space* $(X,\mathcal{O})$ is a nonempty set X with [illegible] without

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two topologies on a set X, then $\mathcal{O}_1$ is said to be *weaker* or *coarser* than $\mathcal{O}_2$ if $\mathcal{O}_1$ is a subset of $\mathcal{O}_2$. Then $\mathcal{O}_2$ is said to be *stronger* or *finer* than $\mathcal{O}_1$ and we write $\mathcal{O}_1 \le \mathcal{O}_2$.

Every metric space X is simultaneously also a topological space. Observe also that different metrics on X can generate equal topologies.

Neighborhoods: Let X be a topological space. A subset U of X is called a *neighborhood*, of the point $a \in X$, if there exists an open set G with $a \in G$ and $G \subset U$. A collection $\mathcal{U}$ of neighborhoods of a point a in X is said to be a *neighborhood basis* of a if for each neighborhood U of a there exists a $V \in \mathcal{U}$ with $V \subset U$.

If $M \subset X$, then $a \in X$ is called an *interior point* of M if there exists a neighborhood U of a with $U \subset M$. The set $\mathring{M}$ of interior points of M is open and is called the *interior* of M.

Closed sets: Let X be a topological space. A subset A of X is said to be *closed* if $X \setminus A$ is open.

If M is a subset of X then the set

$$\overline{M} := X \setminus (X \setminus M)^\circ = \{x \in X : U \cap M \ne \emptyset \text{ for every neighborhood } U \text{ of } x\}$$

is called the *closure* of M. By the above definition $\overline{M}$ is closed. M is closed if, and only if, $M = \overline{M}$ is true.

An element $x \in X$ is called an *adherent point* of M if for each neighborhood U of x, $U \cap M \ne \emptyset$. $\overline{M}$ is also the set of all adherent points of M.

The *boundary* of the set M, denoted ∂M, is defined by

$$\begin{aligned}\partial M &:= \overline{M} \cap (\overline{X \setminus M}) \\ &= \{x \in X : U \cap M \ne \emptyset \ne U \cap (X \setminus M) \text{ for every neighborhood } U \text{ of } x\}\ .\end{aligned}$$

M is said to be *dense* in $N \subset X$, if $\overline{M} \supset N$.

Sequences and nets: Let X be a topological space. A *net* or a *generalized sequence* in X is a family $(x_\alpha)_{\alpha \in A}$ of elements x_α of X, where the index set A is a directed set; in this definition an ordered set $(A,\le)$ is said to be *directed*, if for every pair of elements α,β of A there exists a γ such that $\alpha \le \gamma$ and $\beta \le \gamma$. In order to avoid the set theoretical difficulties we shall pretend that while considering nets, the cardinality of the index set used is below a sufficiently large bound.

A net $(x_\alpha)_{\alpha \in A}$ in X is said to be *convergent* to $x \in X$, if, corresponding to every neighborhood U of x there exists an $\alpha_U \in A$ such that $x_\alpha \in U$ for each $\alpha \in A$ such that $\alpha \ge \alpha_U$. From ($\mathcal{O}$3) it follows that x is uniquely determined and is called the *limit* of the net $(x_\alpha)_{\alpha \in A}$. We denote this by $x = \lim_{\alpha \in A} x_\alpha$ or

If M is a subset of X, then

$$\overline{M} = \{x \in X : \text{there exists a net } (x_\alpha)_{\alpha \in A} \text{ in } M \text{ with } x_\alpha \longrightarrow x\}.$$

Moreover, M is closed if, and only if, for every convergent net of elements of M its limit is also in M.

For a metric space X the description of the closure $\overline{M}$ can be restricted to sequences since, in this case, for every neighborhood U of $a \in X$ there exists an $n \in \mathbb{N}$ such that $U_{\frac{1}{n}}(a) \subset U$.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X,d) converges to x if, and only if, $\lim_{n\to\infty} d(x,x_n) = 0$.

Continuity: Let X and Y be topological spaces and $f : X \longrightarrow Y$ be a mapping. The map f is said to be *continuous at* $a \in X$ if for each neighborhood V of $f(a)$ there exists a neighborhood U of a such that $f(U) \subset V$.

The map f is said to be *continuous* if it is continuous at all points in X. It is called a *homeomorphism* if it is bijective and f and f^{-1} are both continuous.

If X and Y are metric spaces, then f is continuous at $a \in X$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $f(U_\delta(a)) \subset U_\varepsilon(f(a))$.

The following statements on continuous functions are fundamental and are useful.

Lemma 2.1 *Let X and Y be topological spaces. A map $f : X \longrightarrow Y$ is continuous at $a \in X$ if, and only if, for every net $(x_\alpha)_{\alpha\in A}$ converging to a, $(f(x_\alpha))_{\alpha\in A}$ converges to $f(a)$.*

Proposition 2.2 *Let X, Y and Z be topological spaces and let $f : Y \longrightarrow Z$, and $g : X \longrightarrow Y$ be given maps. Then:*

1. *If g is continuous at a and f is continuous at $g(a)$ then $f \circ g$ is continuous at a.*
2. *If f and g are continuous, then so is $f \circ g$.*

Proposition 2.3 *Let X, Y be topological spaces. Then, for $f : X \longrightarrow Y$, the following are equivalent:*

1. *f is continuous.*
2. *$f(\overline{A}) \subset \overline{f(A)}$ for every subset A of X.*
3. *The preimage of every closed set is closed.*
4. *The preimage of every open set is open.*

Remark 2.4 *(a) Let X, Y be topological spaces and M be a subset of X. If the functions $f, g : X \longrightarrow Y$ are continuous and coincide on M then they coincide also on $\overline{M}$.*

... topologies on a set X, then $\mathrm{id}_X : (X, \mathcal{O}_1) \longrightarrow$

Subspaces: Let X be a topological space and $Y \neq \emptyset$ be a subset of X; then, the topology $\mathcal{O}$ of X induces on Y the *induced topology*

$$\widetilde{\mathcal{O}} := \{G \cap Y : G \in \mathcal{O}\}.$$

If the inclusion map is denoted by $j : Y \longrightarrow X$ then it follows from 2.3(4) that $\widetilde{\mathcal{O}}$ is the coarsest topology for which j is continuous.

If (X,d) is a metric space and Y is a nonempty subset of X then $d|_{Y\times Y}$ is a metric on Y, which induces on Y the induced topology.

Topological products: Let $(X_i, \mathcal{O}_i)_{i\in I}$ be a family of topological spaces. We define the canonical map $\pi_i : X \longrightarrow X_i$ on $X := \prod_{i\in I} X_i$ by $\pi_i((x_i)_{i\in I}) := x_i$, $i \in I$. There exists a coarsest topology $\mathcal{O}$ on X for which all the canonical maps are continuous. $\mathcal{O}$ is called the *product topology* on X. In order to describe it, let

$$\mathcal{B} := \left\{ \prod_{i\in I} G_i : G_i \in \mathcal{O}_i \text{ for all } i \in I,\ G_i \neq X_i \text{ only for finitely many } i \in I \right\}.$$

Then we have

$$\mathcal{O} = \{G \subset X : G \text{ is the union of sets from } \mathcal{B}\}.$$

If $(X_j, d_j)_{j=1}^N$ are metric spaces then $X := \prod_{j=1}^N X_j$ will also become a metric space through the product metric

$$d(x,y) := \max_{1\le j\le N} d_j(x_j,y_j), \quad \text{where } x = (x_1,\dots,x_N),\ y = (y_1,\dots,y_N).$$

The metric d on X induces the product topology. Observe also that a sequence in X converges in the product topology if, and only if, its component sequences converge.

If $(X_j, d_j)_{j\in\mathbb{N}}$ is a sequence of metric spaces and $X := \prod_{j=1}^\infty X_j$, then

$$d : X \times X \longrightarrow \mathbb{R}_+,\ d(x,y) := \sum_{j=1}^\infty \frac{1}{2^j} \frac{d_j(x_j,y_j)}{1 + d_j(x_j,y_j)}$$

is a metric on X. Evidently (M1) and (M3) are satisfied, while (M2) follows from the fact that $t \longmapsto \frac{t}{1+t}$ monotonically increases on $[0,\infty[$. It is easy to prove that a sequence $(x^{(n)})_{n\in\mathbb{N}}$ in (X,d) converges to $x^{(0)}$ if, and only if, for all $j \in \mathbb{N}$, $x_j^{(0)} = \lim_{n\to\infty} x_j^{(n)}$. Consequently, d induces the product topology on X.

Remark 2.5 *If X is a metric space, then it follows from (M1) and (M2) that*

$$|d(x,y) - d(\xi,\eta)| \le d(x,\xi) + d(y,\eta) \quad \text{for all } x,y,\xi,\eta \in X.$$

Distance: If A and B are nonempty subsets of the metric space X, then we define the *distance* between A and B as

$$\mathrm{dist}(A,B) := \inf\{d(a,b) : a \in A,\ b \in B\}\ .$$

If $A = \{x\}$, then we write $\mathrm{dist}(x,B)$, instead of, $\mathrm{dist}(\{x\},B)$.

Exercises:

1. Prove 2.1–2.5.
2. Let X be a metric space, $A \neq \emptyset$ be a subset of X. Show:
   (a) $|\mathrm{dist}(x,A) - \mathrm{dist}(y,A)| \leq d(x,y)$.
   (b) $\overline{A} = \{x \in X : \mathrm{dist}(x,A) = 0\}$.
3. Find two disjoint, closed subsets A and B of $\mathbb{R}$ for which $\mathrm{dist}(A,B) = 0$.
4. Let X and $(X_i)_{i \in I}$ be topological spaces. Show that $f : X \longrightarrow \prod_{i \in I} X_i$ is continuous for the product topology if, and only if, $\pi_i \circ f$ is continuous for all $i \in I$.
5. Let (X,d) be a metric space. Show that there exists a metric $\rho : X \times X \longrightarrow [0,1]$ so that d and ρ generate the same topology on X.
6. Prove that the topology of a countable topological product of metric spaces is induced by a metric.
7. Let X be a nonempty set. Show that the function $d : X \times X \longrightarrow \mathbb{R}_+$, $d(x,y) := 0$ for $x = y, d(x,y) = 1$ for $x \neq y$ induces a metric on X. Determine, for $a \in X$ and $\varepsilon > 0$ the following sets in (X,d): $U_\varepsilon(a)$, $\overline{U_\varepsilon(a)}$ as well as $\{x \in X : d(x,a) \leq \varepsilon\}$.

3
Complete metric spaces

In this chapter we shall consider the notion of completeness of metric spaces, a concept which is very important in functional analysis for several reasons.

Definition: A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space is called a *Cauchy sequence* if for each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$, for all $m, n > N$.

It is easy to show that a convergent sequence is a Cauchy sequence. Also, a Cauchy sequence converges if, and only if, it contains a convergent subsequence.

Definition: A metric space X is said to be *complete* if every Cauchy sequence in X is convergent. A subset A of X is said to be complete if it is complete in the induced metric.

Remark 3.1 *Every closed subset of a complete metric space is complete. Every complete subset of a metric space is closed. $\mathbb{R}$ and $\mathbb{C}$ are complete.*

The following theorem, due to Baire, which is very important in functional analysis, is true in complete metric spaces.

Proposition 3.2 *If a complete metric space X is the union of countably many closed subsets M_n, then at least one of the sets M_n has an interior point.*

Proof We shall assume, contrary to the conclusion, that $\mathring{M}_n = \emptyset$ for all $n \in N$, i.e.,

(1) $$U_\varepsilon(x) \cap (X \setminus M_n) \neq \emptyset \text{ for all } x \in X,\ \varepsilon > 0 \text{ and } n \in \mathbb{N}\,.$$

In order to show, then, that $X \neq \bigcup_{n\in\mathbb{N}} M_n$, we choose $x_0 \in X$ and $\varepsilon_0 > 0$ arbitrarily. Then $U_{\varepsilon_0}(x_0) \cap (X \setminus M_1)$ is open and nonempty by (1). Hence there exist $x_1 \in X$ and $0 < \varepsilon_1 < \frac{\varepsilon_0}{2}$ such that

$$U_{2\varepsilon_1}(x_1) \subset U_{\varepsilon_0}(x_0) \cap (X \setminus M_1)\,.$$

Applying this argument inductively we construct a sequence $(x_n)_{n\in\mathbb{N}}$ in X and

(2) $$U_{2\varepsilon_{n+1}}(x_{n+1}) \subset U_{\varepsilon_n}(x_n) \cap (X \setminus M_{n+1}),\ 0 < \varepsilon_{n+1} < \frac{\varepsilon_n}{2}.$$

Given now $m, n \in \mathbb{N}$ with $m > n$, it then follows from (2):

(3) $$d(x_m, x_n) < \varepsilon_n < \frac{\varepsilon_0}{2^n}.$$

Hence $(x_n)_n$ is a Cauchy sequence in X. Then, by hypothesis there exists a $\xi \in X$ with $\xi = \lim_{n\to\infty} x_n$. It then follows from (3) and the continuity of the metric that

$$d(\xi, x_n) = \lim_{m\to\infty} d(x_m, x_n) \le \varepsilon_n < 2\varepsilon_n.$$

From (2) we then have

$$\xi \in U_{2\varepsilon_n}(x_n) \subset X \setminus M_n.$$

Since $n \in \mathbb{N}$ was chosen arbitrarily, we have that $\xi \notin \bigcup_{n\in\mathbb{N}} M_n$. □

Definition: Let X be a metric space and M be a subset of X. M is said to be *nowhere dense* in X if $\overline{M}$ has no interior points. M is said to be of the *I-category* in X if M is the countable union of nowhere dense sets. M is said to be of the *II-category* in X if it is not of the I-category.

With these definitions we now have from Proposition 3.2

Baire category theorem 3.3 *A complete metric space is of the II-category in itself.*

Example 3.4 Let M be a nonempty set. We define

$$l_\infty(M) := \left\{ f : M \longrightarrow \mathbb{K} : \sup_{t\in M} |f(t)| < \infty \right\}.$$

Defining $f + g : t \longmapsto f(t) + g(t)$ and $\lambda f : t \longmapsto \lambda f(t)$ the set $l_\infty(M)$ becomes a $\mathbb{K}$–vector space. By means of the definition

$$d(f,g) := \sup_{t\in M} |f(t) - g(t)|,\ f, g \in l_\infty(M),$$

a metric is defined on $l_\infty(M)$.

$(l_\infty(M), d)$ is complete: let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $l_\infty(M)$. Then we have:

(1) For each $\varepsilon > 0$ there exists an $N = N(\varepsilon) \in \mathbb{N}$, such that for all $n, m \ge N$
$$|f_n(t) - f_m(t)| \le \varepsilon \quad \text{for all } t \in M.$$

Hence, for each $t \in M$, $(f_n(t))_{n\in N}$ is a Cauchy sequence in $\mathbb{K}$ and therefore converges. Thus a function f is defined by $f : t \longmapsto \lim_{n\to\infty} f_n(t)$. It is bounded, since from (1) we have with $\varepsilon = 1$ and $N = N(1)$ for $n \longrightarrow \infty$

$$|f(t) - f_N(t)| \le 1,\ \text{i.e.,}\ |f(t)| \le 1 + |f_N(t)| \text{ for all } t \in M.$$

... (1) that for each $\varepsilon > 0$ and every $m > N(\varepsilon)$, $d(f, f_m) \le \varepsilon$.

Definition: Let X and Y be metric spaces. A map $f : X \longrightarrow Y$ is said to be *uniformly continuous* if, for each given $\varepsilon > 0$, there exists a $\delta > 0$, such that $d(f(x), f(y)) < \varepsilon$ for all $x, y \in X$ satisfying $d(x,y) < \delta$.

Clearly, every uniformly continuous map is continuous; the converse is, however, not true. Every uniformly continuous map maps a Cauchy sequence into a Cauchy sequence. Also, from (2.5) it follows that for a metric space X the metric
$d : X \times X \longrightarrow \mathbb{R}_+$ is uniformly continuous.

Lemma 3.5 *Let X and Y be metric spaces and A be a dense subset of X and let Y be complete. If $f : A \longrightarrow Y$ is uniformly continuous then there exists a unique continuous map $F : X \longrightarrow Y$, with $F|_A = f$. F is also uniformly continuous.*

Proof Since A is dense in X, for each $x \in X$ there exists a sequence $(a_n)_{n\in\mathbb{N}}$ converging to x. Since f is uniformly continuous on A, $(f(a_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Because Y is complete, we can define $F(x) := \lim_{n\to\infty} f(a_n)$ and this definition does not depend on the sequence $(a_n)_{n\in\mathbb{N}}$. Indeed, if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ both converge to x, then $\lim_{n\to\infty} d(a_n, b_n) = 0$ and therefore $(f(a_n))_{n\in\mathbb{N}}$ and $(f(b_n))_{n\in\mathbb{N}}$ have the same limit. Thus we obtain a map $F : X \longrightarrow Y$ with $F|_A = f$.

In order to prove the uniform continuity of F, let $\varepsilon > 0$ be given. Since f is uniformly continuous on A, there exists a $\delta > 0$ such that for all $a, b \in A$ with $d(a,b) < \delta$ we have $d(f(a), f(b)) < \varepsilon$. Now for given $x, y \in X$ with $d(x,y) < \delta$, choose sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in A, converging respectively to x and y. Then, for sufficiently large $n \in \mathbb{N}$ we have

$$d(a_n, b_n) \le d(a_n, x) + d(x, y) + d(y, b_n) < \delta$$

and thus $d(f(a_n), f(b_n)) < \varepsilon$. Since $F(x) = \lim_{n\to\infty} f(a_n)$, $F(y) = \lim_{n\to\infty} f(b_n)$, the choice of δ implies $d(F(x), F(y)) \le \varepsilon$ since for each $n \in \mathbb{N}$:

$$d(F(x), F(y)) \le d(F(x), f(a_n)) + d(f(a_n), f(b_n)) + d(f(b_n), F(y)).$$

The proof of the uniqueness follows immediately from 2.4(a). □

Definition: A map $f : X \longrightarrow Y$ between metric spaces is called an *isometry* if for all $x, y \in X$, $d(f(x), f(y)) = d(x,y)$.

Every isometry is uniformly continuous.

Definition: Let X be a metric space. A *completion* of X is a pair $(\widehat{X}, j)$, consisting of a complete metric space $\widehat{X}$ and an isometry $j : X \longrightarrow \widehat{X}$, for which

If $(\widehat{X}, j)$ is a completion of the metric space X then X is usually identified with $j(X)$ by means of the isometry j; i.e., X is viewed as a subspace of $\widehat{X}$. Moreover, we do not refer to $\widehat{X}$ as a completion of X but as *the* completion of X. This is justified by the following lemma.

Lemma 3.6 *Suppose that (X_1, j_1) and (X_2, j_2) are two completions of the metric space X. Then there exists a bijective isometry $J : X_1 \longrightarrow X_2$ with $J \circ j_1 = j_2$.*

Proof Let $A := j_1(X) \subset X_1$ and define $j : A \longrightarrow X_2$ by setting $j(j_1(x)) = j_2(x)$. Then j is an isometry and therefore uniformly continuous. Since A is dense in X_1 it follows from 3.5 that there exists a unique continuous map $J : X_1 \longrightarrow X_2$ such that $J|_A = j$, i.e., $J \circ j_1 = j_2$.

J is an isometry: for $x, y \in X_1$ and sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in A such that $x_n \longrightarrow x$, $y_n \longrightarrow y$ we have, from 2.5 :

$$d(J(x), J(y)) = \lim_{n\to\infty} d(j(x_n), j(y_n)) = \lim_{n\to\infty} d(x_n, y_n) = d(x, y)\,.$$

Since J is an isometry and X_1 is complete, the range of J is closed and since $\operatorname{Im} J$ contains the dense set $j_2(X)$, we see that J is surjective. □

Proposition 3.7 *Every metric space X has a completion $\widehat{X}$. It has the following properties: if Y is a complete metric space and $f : X \longrightarrow Y$ is a uniformly continuous map then there exists a unique, uniformly continuous map $F : \widehat{X} \longrightarrow Y$ such that $F|_X = f$.*

Proof Let $a \in X$ be fixed. For $x \in X$ define $j(x) : X \longrightarrow \mathbb{R}$ by setting $j(x) : t \longmapsto d(x,t) - d(t,a)$. From the triangle inequality we have

$$|j(x)[t]| = |d(x,t) - d(t,a)| \le d(x,a) \text{ for all } t \in X.$$

Hence $j(x) \in l_\infty(X)$ for each $x \in X$. The map $j : X \longrightarrow l_\infty(X)$ is an isometry as the following two estimates show:

$$d(j(x), j(y)) = \sup_{t\in X} |d(x,t) - d(t,y)| \le d(x,y),$$
$$d(j(x), j(y)) \ge |d(x,y) - d(y,y)| = d(x,y)\,.$$

If we now set $\widehat{X} := \overline{j(X)}$, then $\widehat{X}$, as a closed subset of the complete metric space $l_\infty(X)$, is complete in the induced metric. Thus $(\widehat{X}, j)$ is a completion of X; that $\widehat{X}$ has the given properties follows from 3.5. □

Remark 3.8 *If $X_1, \ldots, X_m$ are metric spaces, then a sequence, $(x^{(n)})_{n\in\mathbb{N}}$, $x^{(n)} = (x_1^{(n)}, \ldots, x_m^{(n)})$, in $X := X_1 \times \ldots \times X_m$ is a Cauchy sequence in the product metric if, and only if, for $1 \le j \le m$ the sequences $(x_j^{(n)})_{n\in\mathbb{N}}$ are Cauchy sequences in X_j. Thus the completion of $X_1 \times \ldots \times X_m$ can be identified with $(\widehat{X}_1 \times \ldots \times$*

As a further application of the notion of completeness we prove a lemma on open maps which will be used in Chapter 8.

Definition: Let X and Y be topological spaces. A map $f : X \longrightarrow Y$ is said to be *open* if $f(U)$ is open for every open set $U \subset X$.

If X and Y are metric spaces, then $f : X \longrightarrow Y$ is open if, and only if, for every $x \in X$ and each $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(U_\varepsilon(x)) \supset U_\delta(f(x))$.

Injective maps are open if, and only if, $f(X)$ is open and the inverse $f^{-1} : f(X) \longrightarrow X$ is continuous.

Lemma 3.9 *Let X and Y be metric spaces; X be complete. Let $f : X \longrightarrow Y$ be continuous and assume that*

(1) *for every $\varepsilon > 0$ there exists a $\delta > 0$, such that for all $x \in X$,*
$$\overline{f(U_\varepsilon(x))} \supset U_\delta(f(x))\,.$$

Then the map f is open.

Proof From the previous remark it follows that it is enough to show that

(2) for each $\varepsilon > 0$ there is $\delta_1 > 0$ such that for $x \in X$: $f(U_\varepsilon(x)) \supset U_{\delta_1}(f(x))$.

To prove (2), let $\varepsilon > 0$ be given. Then we set $\varepsilon_n := \frac{\varepsilon}{2^n}$ for $n \in \mathbb{N}$ and determine, according to (1), for ε_n a δ_n with $\delta_n \le \frac{1}{n}$. If $x \in X$ is fixed and $y \in U_{\delta_1}(f(x))$ is arbitrary, then we choose inductively a sequence $(x_n)_{n\in\mathbb{N}_0}$ in X with $x = x_0$ and

(3) $d(f(x_n), y) < \delta_{n+1} \le \frac{1}{n+1}$ and $d(x_n, x_{n-1}) < \varepsilon_n = \frac{\varepsilon}{2^n}$ for $n \in \mathbb{N}$.

In order to achieve the above construction we use the following scheme: assume that x_n has been chosen with $d(f(x_n), y) < \delta_{n+1}$; then, we have, from (1) and (3):

$$y \in U_{\delta_{n+1}}(f(x_n)) \subset \overline{f(U_{\varepsilon_{n+1}}(x_n))} \subset \bigcup_{\xi\in U_{\varepsilon_{n+1}}(x_n)} U_{\delta_{n+2}}(f(\xi))\,.$$

Hence there exists an $x_{n+1} \in U_{\varepsilon_{n+1}}(x_n)$, such that $y \in U_{\delta_{n+2}}(f(x_{n+1}))$, i.e.,

$$d(f(x_{n+1}), y) < \delta_{n+2} \text{ and } d(x_{n+1}, x_n) < \varepsilon_{n+1}\,.$$

From (3), we have that $(x_n)_{n\in\mathbb{N}_0}$ is a Cauchy sequence in X. Since X is complete, there exists $\xi := \lim_{n\to\infty} x_n$, and from (3) it follows that

$$d(x,\xi) = d(x_0,\xi) = \lim_{k\to\infty} d(x_0, x_k) \le \lim_{k\to\infty} \sum_{n=1}^{k} d(x_n, x_{n-1}) < \sum_{n=1}^{\infty} \varepsilon 2^{-n} = \varepsilon\,,$$

i.e., $\xi \in U_\varepsilon(x)$. By the continuity of f, it follows from (3) that

$$y = \lim_{n\to\infty} f(x_n) = f(\xi)\,.$$

Exercises:

1. For $j \in \mathbb{N}$ let $e_j \in \ell_\infty(\mathbb{N})$ be the sequence $e_j = (\delta_{j,k})_{k\in\mathbb{N}}$. Show that the closure of $\varphi = \{x \in \ell_\infty(\mathbb{N}) : x_j = 0 \quad \text{for almost all } j \in \mathbb{N}\}$ is the space c_0 of all null sequences.
2. Let $C_c(\mathbb{R}) := \{f \in \ell_\infty(\mathbb{R}) : f$ is continuous and $f|_{\mathbb{R}\setminus[-n,n]} \equiv 0$ for an $n = n(f)\}$. Show that the closure of $C_c(\mathbb{R})$ in $\ell_\infty(\mathbb{R})$ is the following set:

$$C_0(\mathbb{R}) = \{f : \mathbb{R} \longrightarrow \mathbb{K} : f \quad \text{is continuous and for every } \varepsilon > 0 \text{ there exists an } n \in \mathbb{N} \text{ such that } \sup_{|x|>n} |f(x)| \le \varepsilon\}.$$

3. Let $C[0,1] := \{f : [0,1] \longrightarrow \mathbb{K} : f \quad \text{is continuous}\}$. Show:
   (a) $d(f,g) := \int_0^1 \frac{|f(t)-g(t)|}{1+|f(t)-g(t)|}\, dt$ defines a metric on $C[0,1]$.
   (b) $(C[0,1], d)$ is not complete.
4. Let X be a complete metric space. Show:
   (a) If $M \subset X$ is of the I-category, then $X\setminus M$ is dense in X.
   (b) If $(G_n)_{n\in\mathbb{N}}$ is a sequence of open, dense sets in X, then $\bigcap_{n\in\mathbb{N}} G_n$ is dense in X.
5. Let X be a complete metric space. Let M be a family of continuous functions on X such that $\sup_{f\in M} |f(x)| < \infty$ for every $x \in X$. Show that there exist an $x_0 \in X$ and an $\varepsilon > 0$ such that

$$\sup\{|f(x)| : f \in M, x \in U_\varepsilon(x_0)\} < \infty\,.$$

6. Prove the following fixed point theorem due to Banach: Let X be a complete metric space and $T : X \longrightarrow X$ be a contraction map, i.e., there exists $0 < q < 1$ such that $(Tx, Ty) \le q d(x,y)$ for all $x, y \in X$. Then there exists a unique $z \in X$ such that $Tz = z$, i.e., T has a unique fixed point.
7. Let $d, \rho : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}_+$ be defined as follows:

$$d(x,y) := |x-y|\,, \quad \rho(x,y) := \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right|\,.$$

   Prove the following statements:
   (a) ρ is a metric on $\mathbb{R}$ and induces the same topology as d.
   (b) There exists a sequence which is Cauchy in $(\mathbb{R}, \rho)$, but is not Cauchy in $(\mathbb{R}, d)$.
   (c) $(\mathbb{R}, \rho)$ is not complete.

4 Compactness

In this chapter we shall consider the notion of compactness in topological spaces. In particular, we shall characterize compact subsets in metric spaces. Also, for later application, we will present the Arzelà–Ascoli, Stone–Weierstrass and Tychonoff theorems. The reader could skip this chapter and return to it when needed.

Definition A topological space X is said to be *compact*, if each of its open coverings contains a finite subcovering. A subset X of a topological space Y is said to be compact if X is compact in the induced topology. This is obviously equivalent to the following: if I is an arbitrary index set, $(G_i)_{i\in I}$ is a family of open subsets of Y and $X \subset \bigcup_{i\in I} G_i$, then there exists a finite subcollection J of I such that $X \subset \bigcup_{i\in J} G_i$.

Remark 4.1 *(a) A topological space X is compact if, and only if, for each family $(A_i)_{i\in I}$ of closed subsets A_i of X such that $\bigcap_{i\in I} A_i = \emptyset$, there exists a finite subset J of I such that $\bigcap_{i\in J} A_i = \emptyset$. This follows by considering the corresponding complementary sets.*

(b) If Y is a topological space and X is a compact subset of Y, then X is closed. If $y \in Y\setminus X$, then corresponding to each $x \in X$, we have from (O3) that there are open sets U_x and V_x such that $U_x \cap V_x = \emptyset$ and $x \in U_x$, $y \in V_x$. Since X is compact, there exist $x_1, \ldots, x_n \in X$ with $X \subset \bigcup_{j=1}^n U_{x_j}$. If we now set $W := \bigcap_{j=1}^n V_{x_j}$, then W is a neighborhood of y and $W \cap X = \emptyset$. Consequently $Y\setminus X$ is open, i.e., X is closed.

(c) Every closed subset X of a compact topological space Y is compact. This follows from the observation that by including $Y\setminus X$ to any open covering of X one obtains an open covering of Y.

Proposition 4.2 *Let X and Y be topological spaces and let $f : X \longrightarrow Y$ be a continuous map. Then:*

1. *If X is compact then so is $f(X)$.*
2. *If X is compact and f is bijective, then f is a homeomorphism.*

Proof (1) If $(G_i)_{i\in I}$ is an open covering of $f(X)$, then, from 2.3 it follows that $(f^{-1}(G_i))_{i\in I}$ is an open covering of X. Since X is compact, it has a finite