

is nonlinear for $n \neq 0$ or 1. Show that the substitution $u = y^{1-n}$ reduces Bernoulli's equation to a linear equation. (See Chapter 18.)

$$\text{ANS. } \frac{du}{dx} + (1-n)f(x)u = (1-n)g(x).$$

- 8.2.17** Solve the linear, first-order equation (8.25) by assuming $y(x) = u(x)v(x)$, where $v(x)$ is a solution of the corresponding homogeneous equation [$q(x) = 0$]. This is the method of **variation of parameters** due to Lagrange. We apply it to second-order equations in Exercise 8.6.25.

8.3 SEPARATION OF VARIABLES

The equations of mathematical physics listed in Section 8.1 are all partial differential equations. Our first technique for their solution splits the partial differential equation of n variables into n ordinary differential equations. Each separation introduces an arbitrary constant of separation. If we have n variables, we have to introduce $n - 1$ constants, determined by the conditions imposed in the problem being solved.

Cartesian Coordinates

In Cartesian coordinates the Helmholtz equation becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0, \quad (8.34)$$

using Eq. (2.26) for the Laplacian. For now let k^2 be a constant. Perhaps the simplest way of treating a partial differential equation such as (8.34) is to split it into a set of ordinary differential equations. This may be done as follows. Let

$$\psi(x, y, z) = X(x)Y(y)Z(z) \quad (8.35)$$

and substitute back into Eq. (8.34). How do we know Eq. (8.35) is valid? We are proceeding in the spirit of let us try and see if it works. If our attempt succeeds, then Eq. (8.35) will be justified. If it does not succeed, we shall find out soon enough and then we shall try another attack such as Green's functions, integral transforms, or brute force numerical analysis. With ψ assumed given by Eq. (8.35), Eq. (8.34) becomes

$$YZ \frac{d^2 X}{dx^2} + XZ \frac{d^2 Y}{dy^2} + XY \frac{d^2 Z}{dz^2} + k^2 XYZ = 0. \quad (8.36)$$

Dividing by $\psi = XYZ$ and rearranging terms, we obtain

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -k^2 - \frac{1}{Y} \frac{d^2 Y}{dy^2} - \frac{1}{Z} \frac{d^2 Z}{dz^2}. \quad (8.37)$$

Equation (8.37) exhibits one separation of variables. The left-hand side is a function of x alone, whereas the right-hand side depends only on y and z , but x , y , and z are all independent coordinates. This independence means that the behavior of x as an independent variable is not determined by y and z . Therefore, each side must be equal to a constant, a constant of separation. We choose³

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -l^2, \quad (8.38)$$

$$-k^2 - \frac{1}{Y} \frac{d^2 Y}{dy^2} - \frac{1}{Z} \frac{d^2 Z}{dz^2} = -l^2. \quad (8.39)$$

Now, turning our attention to Eq. (8.39), we obtain

$$\frac{1}{Y} \frac{d^2 Y}{dy^2} = -k^2 + l^2 - \frac{1}{Z} \frac{d^2 Z}{dz^2}, \quad (8.40)$$

and a second separation has been achieved. Here we have a function of y equated to a function of z as before. We resolve it as before by equating each side to another constant of separation³ $-m^2$,

$$\frac{1}{Y} \frac{d^2 Y}{dy^2} = -m^2, \quad (8.41)$$

$$\frac{1}{Z} \frac{d^2 Z}{dz^2} = -k^2 + l^2 + m^2 = -n^2, \quad (8.42)$$

introducing a constant n^2 by $k^2 = l^2 + m^2 + n^2$ to produce a symmetric set of equations. Now we have three ODEs ((8.38), (8.41), and (8.42)) to replace Eq. (8.34). Our assumption (Eq. (8.35)) has succeeded and is thereby justified.

Our solution should be labeled according to the choice of our constants l , m , and n , that is,

$$\psi_{lmn}(x, y, z) = X_l(x)Y_m(y)Z_n(z). \quad (8.43)$$

Subject to the conditions of the problem being solved and to the condition $k^2 = l^2 + m^2 + n^2$, we may choose l , m , and n as we like, and Eq. (8.43) will still be a solution of Eq. (8.34), provided $X_l(x)$ is a solution of Eq. (8.38), and so on. We may develop **the most general solution** of Eq. (8.34) by taking a **linear combination of solutions** ψ_{lmn} ,

$$\Psi = \sum_{l,m,n} a_{lmn} \psi_{lmn}. \quad (8.44)$$

³The choice of sign, completely arbitrary here, will be fixed in specific problems by the need to satisfy specific boundary conditions.

The constant coefficients a_{lmn} are finally chosen to permit Ψ to satisfy the boundary conditions of the problem.

Circular Cylindrical Coordinates

With our unknown function ψ dependent on ρ , φ , and z , the Helmholtz equation becomes (see Section 2.4 for ∇^2)

$$\nabla^2 \psi(\rho, \varphi, z) + k^2 \psi(\rho, \varphi, z) = 0, \quad (8.45)$$

or

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0. \quad (8.46)$$

As before, we assume a factored form for ψ ,

$$\psi(\rho, \varphi, z) = P(\rho)\Phi(\varphi)Z(z). \quad (8.47)$$

Substituting into Eq. (8.46), we have

$$\frac{\Phi Z}{\rho} \frac{d}{d\rho} \left(\rho \frac{dP}{d\rho} \right) + \frac{PZ}{\rho^2} \frac{d^2 \Phi}{d\varphi^2} + P\Phi \frac{d^2 Z}{dz^2} + k^2 P\Phi Z = 0. \quad (8.48)$$

All the partial derivatives have become ordinary derivatives. Dividing by $P\Phi Z$ and moving the z derivative to the right-hand side yields

$$\frac{1}{\rho P} \frac{d}{d\rho} \left(\rho \frac{dP}{d\rho} \right) + \frac{1}{\rho^2 \Phi} \frac{d^2 \Phi}{d\varphi^2} + k^2 = -\frac{1}{Z} \frac{d^2 Z}{dz^2}. \quad (8.49)$$

Again, a function of z on the right appears to depend on a function of ρ and φ on the left. We resolve this by setting each side of Eq. (8.49) equal to the same constant. Let us choose⁴ $-l^2$. Then

$$\frac{d^2 Z}{dz^2} = l^2 Z, \quad (8.50)$$

and

$$\frac{1}{\rho P} \frac{d}{d\rho} \left(\rho \frac{dP}{d\rho} \right) + \frac{1}{\rho^2 \Phi} \frac{d^2 \Phi}{d\varphi^2} + k^2 = -l^2. \quad (8.51)$$

Setting $k^2 + l^2 = n^2$, multiplying by ρ^2 , and rearranging terms, we obtain

$$\frac{\rho}{P} \frac{d}{d\rho} \left(\rho \frac{dP}{d\rho} \right) + n^2 \rho^2 = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2}. \quad (8.52)$$

⁴The choice of sign of the separation constant is arbitrary. However, a minus sign is chosen for the axial coordinate z in expectation of a possible exponential dependence on z (from Eq. (8.50)). A positive sign is chosen for the azimuthal coordinate φ in expectation of a periodic dependence on φ (from Eq. (8.53)).

We may set the right-hand side to m^2 and

$$\frac{d^2\Phi}{d\varphi^2} = -m^2\Phi. \quad (8.53)$$

Finally, for the ρ dependence we have

$$\rho \frac{d}{d\rho} \left(\rho \frac{dP}{d\rho} \right) + (n^2\rho^2 - m^2)P = 0. \quad (8.54)$$

This is Bessel's differential equation. The solutions and their properties are presented in Chapter 11. The separation of variables of Laplace's equation in parabolic coordinates also gives rise to Bessel's equation. It may be noted that the Bessel equation is notorious for the variety of disguises it may assume. For an extensive tabulation of possible forms the reader is referred to *Tables of Functions* by Jahnke and Emde.⁵

The original Helmholtz equation, a three-dimensional PDE, has been replaced by three ODEs, Eqs. (8.50), (8.53), and (8.54). A solution of the Helmholtz equation is

$$\psi(\rho, \varphi, z) = P(\rho)\Phi(\varphi)Z(z). \quad (8.55)$$

Identifying the specific P , Φ , Z solutions by subscripts, we see that the most general solution of the Helmholtz equation is a linear combination of the product solutions:

$$\Psi(\rho, \varphi, z) = \sum_{m,n} a_{mn} P_{mn}(\rho)\Phi_m(\varphi)Z_n(z). \quad (8.56)$$

Spherical Polar Coordinates

Let us try to separate the Helmholtz equation, again with k^2 constant, in spherical polar coordinates. Using Eq. (2.46), we obtain

$$\frac{1}{r^2 \sin \theta} \left[\sin \theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 \psi}{\partial \varphi^2} \right] = -k^2 \psi. \quad (8.57)$$

Now, in analogy with Eq. (8.35) we try

$$\psi(r, \theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi). \quad (8.58)$$

By substituting back into Eq. (8.57) and dividing by $R\Theta\Phi$, we have

$$\frac{1}{Rr^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta r^2 \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\Phi r^2 \sin^2 \theta} \frac{d^2\Phi}{d\varphi^2} = -k^2. \quad (8.59)$$

⁵E. Jahnke and F. Emde, *Tables of Functions*, 4th rev. ed., New York: Dover (1945), p. 146; also, E. Jahnke, F. Emde, and F. Lösch, *Tables of Higher Functions*, 6th ed., New York: McGraw-Hill (1960).

Note that all derivatives are now ordinary derivatives rather than partials. By multiplying by $r^2 \sin^2 \theta$, we can isolate $(1/\Phi)(d^2\Phi/d\varphi^2)$ to obtain⁶

$$\frac{1}{\Phi} \frac{d^2\Phi}{d\varphi^2} = r^2 \sin^2 \theta \left[-k^2 - \frac{1}{r^2 R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) - \frac{1}{r^2 \sin \theta \Theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) \right]. \quad (8.60)$$

Equation (8.60) relates a function of φ alone to a function of r and θ alone. As r , θ , and φ are independent variables, we equate each side of Eq. (8.60) to a constant. In almost all physical problems φ will appear as an azimuth angle. This suggests a periodic solution rather than an exponential. With this in mind, let us use $-m^2$ as the separation constant. Then

$$\frac{1}{\Phi} \frac{d^2\Phi(\varphi)}{d\varphi^2} = -m^2 \quad (8.61)$$

and

$$\frac{1}{r^2 R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{1}{r^2 \sin \theta \Theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) - \frac{m^2}{r^2 \sin^2 \theta} = -k^2. \quad (8.62)$$

Multiplying Eq. (8.62) by r^2 and rearranging terms, we obtain

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + r^2 k^2 = -\frac{1}{\sin \theta \Theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \frac{m^2}{\sin^2 \theta}. \quad (8.63)$$

Again, the variables are separated. We equate each side to a constant Q and finally obtain

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} \Theta + Q\Theta = 0, \quad (8.64)$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + k^2 R - \frac{QR}{r^2} = 0. \quad (8.65)$$

Once more we have replaced a partial differential equation of three variables by three ODEs. The solutions of these ODEs are discussed in Chapters 11 and 12. In Chapter 12, for example, Eq. (8.64) is identified as the associated Legendre equation in which the constant Q becomes $l(l+1)$; l is an integer. If k^2 is a (positive) constant, Eq. (8.65) becomes the spherical Bessel equation of Section 11.7.

Again, our most general solution may be written

$$\psi_{Qm}(r, \theta, \varphi) = \sum_{Q,m} a_{Qm} R_Q(r) \Theta_{Qm}(\theta) \Phi_m(\varphi). \quad (8.66)$$

The restriction that k^2 be a constant is unnecessarily severe. The separation process will still be possible for k^2 as general as

⁶The order in which the variables are separated here is not unique. Many quantum mechanics texts show the r dependence split off first.

$$k^2 = f(r) + \frac{1}{r^2}g(\theta) + \frac{1}{r^2 \sin^2 \theta}h(\varphi) + k'^2. \quad (8.67)$$

In the hydrogen atom problem, one of the most important examples of the Schrödinger wave equation with a closed form solution is $k^2 = f(r)$. Equation (8.65) for the hydrogen atom becomes the associated Laguerre equation.

The great importance of this separation of variables in spherical polar coordinates stems from the fact that the case $k^2 = k^2(r)$ covers a tremendous amount of physics: a great deal of the theories of gravitation, electrostatics, atomic, nuclear and particle physics. With $k^2 = k^2(r)$, the angular dependence is isolated in Eqs. (8.61) and (8.64), **which can be solved exactly**.

Finally, as an illustration of how the constant m in Eq. (8.61) is restricted, we note that φ in cylindrical and spherical polar coordinates is an azimuth angle. If this is a classical problem, we shall certainly require that the azimuthal solution $\Phi(\varphi)$ be single-valued, that is,

$$\Phi(\varphi + 2\pi) = \Phi(\varphi). \quad (8.68)$$

This is equivalent to requiring the azimuthal solution to have a period of 2π or some integral multiple of it.⁷ Therefore m must be an integer. Which integer it is depends on the details of the problem. This is discussed in Chapter 9. Whenever a coordinate corresponds to an axis of translation or to an azimuth angle the separated equation always has the form

$$\frac{d^2\Phi(\varphi)}{d\varphi^2} = -m^2\Phi(\varphi)$$

for φ , the azimuth angle, and

$$\frac{d^2Z(z)}{dz^2} = \pm a^2Z(z) \quad (8.69)$$

for z , an axis of translation in one of the cylindrical coordinate systems. The solutions, of course, are $\sin az$ and $\cos az$ for $-a^2$ and the corresponding hyperbolic function (or exponentials) $\sinh az$ and $\cosh az$ for $+a^2$.

Other occasionally encountered ODEs include the Laguerre and associated Laguerre equations from the supremely important hydrogen atom problem in quantum mechanics:

$$x \frac{d^2y}{dx^2} + (1-x) \frac{dy}{dx} + \alpha y = 0, \quad (8.70)$$

$$x \frac{d^2y}{dx^2} + (1+k-x) \frac{dy}{dx} + \alpha y = 0, \quad (8.71)$$

⁷This also applies in most quantum mechanical problems but the argument is much more involved. If m is not an integer, rotation group relations and ladder operator relations (Section 4.3) are disrupted. Compare E. Merzbacher, Single valuedness of wavefunctions. *Am. J. Phys.* **30**: 237 (1962).

From the quantum mechanical theory of the linear oscillator we have Hermite's equation,

$$\frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2\alpha y = 0, \quad (8.72)$$

Finally, from time to time we find the Chebyshev differential equation

$$(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + n^2 y = 0. \quad (8.73)$$

For convenient reference, the forms of the solutions of Laplace's equation, Helmholtz's equation, and the diffusion equation for spherical polar coordinates are collected in Table 8.2. The solutions of Laplace's equation in circular cylindrical coordinates are presented in Table 8.3.

Table 8.2 Solutions in Spherical Polar Coordinates^a

| $\psi = \sum_{l,m} a_{lm} \psi_{lm}$ | | |
|--------------------------------------|--------------------------------|--|
| 1. | $\nabla^2 \psi = 0$ | $\psi_{lm} = \begin{Bmatrix} r^l \\ r^{-l-1} \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos \theta) \\ Q_l^m(\cos \theta) \end{Bmatrix} \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix}^b$ |
| 2. | $\nabla^2 \psi + k^2 \psi = 0$ | $\psi_{lm} = \begin{Bmatrix} j_l(kr) \\ n_l(kr) \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos \theta) \\ Q_l^m(\cos \theta) \end{Bmatrix} \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix}^b$ |
| 3. | $\nabla^2 \psi - k^2 \psi = 0$ | $\psi_{lm} = \begin{Bmatrix} i_l(kr) \\ k_l(kr) \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos \theta) \\ Q_l^m(\cos \theta) \end{Bmatrix} \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix}^b$ |

^a References for some of the functions are $P_l^m(\cos \theta)$, $m = 0$, Section 12.1; $m \neq 0$, Section 12.5; and $Q_l^m(\cos \theta)$, Section 12.10; $j_l(kr)$, $n_l(kr)$, $i_l(kr)$, and $k_l(kr)$, Section 11.7.

^b $\cos m\varphi$ and $\sin m\varphi$ may be replaced by $e^{\pm im\varphi}$.

Table 8.3 Solutions in Circular Cylindrical Coordinates^a

| $\psi = \sum_{m,\alpha} a_{m\alpha} \psi_{m\alpha}, \quad \nabla^2 \psi = 0$ | | |
|--|--------------------------------------|---|
| a. | | $\psi_{m\alpha} = \begin{Bmatrix} J_m(\alpha\rho) \\ N_m(\alpha\rho) \end{Bmatrix} \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix} \begin{Bmatrix} e^{-\alpha z} \\ e^{\alpha z} \end{Bmatrix}$ |
| b. | | $\psi_{m\alpha} = \begin{Bmatrix} I_m(\alpha\rho) \\ K_m(\alpha\rho) \end{Bmatrix} \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix} \begin{Bmatrix} \cos \alpha z \\ \sin \alpha z \end{Bmatrix}$ |
| c. | If $\alpha = 0$ (no z -dependence) | $\psi_m = \begin{Bmatrix} \rho^m \\ \rho^{-m} \end{Bmatrix} \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix}$ |

^aReferences for the radial functions are $J_m(\alpha\rho)$, Section 11.1; $N_m(\alpha\rho)$, Section 11.3; and $I_m(\alpha\rho)$ and $K_m(\alpha\rho)$, Section 11.5.

For the Helmholtz and the diffusion equation the constant $\pm k^2$ is added to the separation constant $\pm\alpha^2$ to define a new parameter γ^2 or $-\gamma^2$. For the choice $+\gamma^2$ (with $\gamma^2 > 0$) we get $J_m(\gamma\rho)$ and $N_m(\gamma\rho)$. For the choice $-\gamma^2$ (with $\gamma^2 > 0$) we get $I_m(\gamma\rho)$ and $K_m(\gamma\rho)$ as previously.

These ODEs and two generalizations of them will be examined and systematized in the next section. General properties following from the form of the differential equations are discussed in Chapter 9. The individual solutions are developed and applied in Chapters 10–13.

The practicing physicist may and probably will meet other second-order ODEs, some of which may possibly be transformed into the examples studied here. Some of these ODEs may be solved by the techniques of Sections 8.5 and 8.6. Others may require a computer for a numerical solution.

We refer to the second edition of this text for other important coordinate systems.

- To put the separation method of solving PDEs in perspective, let us review it as a consequence of a symmetry of the PDE. Take the stationary Schrödinger equation $H\psi = E\psi$ as an example with a potential $V(r)$ depending only on the radial distance r . Then this PDE is invariant under rotations that comprise the group $SO(3)$. Its generator is the orbital angular momentum operator $L_z = -i(\partial/\partial\varphi)$, and its quadratic (Casimir) invariant \mathbf{L}^2 . As both commute with H (see Section 4.3), we end up with three separate eigenvalue equations

$$H\psi = E\psi, \quad \mathbf{L}^2\psi = l(l+1)\psi, \quad L_z\psi = m\psi.$$

Upon replacing L_z^2 in \mathbf{L}^2 by its eigenvalue m^2 , the \mathbf{L}^2 PDE becomes Legendre's ODE and similarly $H\psi = E\psi$ becomes the radial ODE of the separation method in spherical polar coordinates.

- For cylindrical coordinates the PDE is invariant under rotations about the z -axis only, which form a subgroup of $SO(3)$. This invariance yields the generator $L_z = -i\partial/\partial\varphi$ and separate azimuthal ODE $L_z\psi = m\psi$ as before. Invariance under translations along the z -axis with the generator $-i\partial/\partial z$ gives the separate ODE in the z variable.
- In general (see Section 4.3), there are n mutually commuting generators H_i with eigenvalues m_i of the (classical) Lie group G of rank n and the corresponding Casimir invariants C_i with eigenvalues c_i , which yield the separate ODEs

$$H_i\psi = m_i\psi, \quad C_i\psi = c_i\psi$$

in addition to the (by now) radial ODE $H\psi = E\psi$.