Norms

Definition : (Norm)

Normed linear space
$$\begin{cases} ① \ \|x\| \geq 0 \quad \& \quad \|x\| = 0 \ \text{iff} \ x = 0 \\ ② \ \|\alpha x\| = |\alpha| \, \|x\|, \quad \alpha \in \mathbb{R} \\ ③ \ \|x+y\| \leq \|x\| + \|y\| \ \text{or} \ \big| \|x\| - \|y\| \big| \leq \|x-y\| \end{cases}$$

Kinds :

- Maximum Norm (row Norm) : $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$
- 1-norm (column norm) : $\|x\|_1 = \sum_{i=1}^{n} |x_i|$
- 2-norm : $\|x\|_2 = \sqrt{\sum_{i=1}^{n} (x_i)^2}$
- P-norm : $\|x\|_p = \left[\sum_{i=1}^{n} (x_i)^p \right]^{1/p}$

  ↳ Not easy to show to satisfy ③

Example : Vector $x = (1, 0, -1, 2)$,

$\|x\|_1 = 4$, $\|x\|_2 = \sqrt{6}$, $\|x\|_\infty = 2$

Example : Matrix $A = \begin{pmatrix} 2 & 3 \\ -4 & 5 \end{pmatrix}$

Remark : inner product space

① $\langle x, y \rangle = \langle y, x \rangle$

② $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

③ $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$

④ $\langle x, x \rangle \geq 0$, & $\langle x, x \rangle = 0$ iff $x = 0$

The inner product space contains angle information but the normed space does NOT.

Thm : All norms in $\mathbb{R}^n$ are equivalent.

3 Equivalence of two norms

Let $\|\cdot\|_{(1)}$ & $\|\cdot\|_{(2)}$ be two norms, i.e. $\|\cdot\|_{(1)} \sim \|\cdot\|_{(2)}$

If $\exists$ c_1 & $c_2 > 0$ $\ni$ $\forall x \in \mathbb{R}^n$

$$c_1 \|x\|_{(1)} \le \|x\|_{(2)} \le c_2 \|x\|_{(1)}$$

where c_1 & c_2 are independent on x

Example: Find c_1 & c_2 $\ni$

$$c_1 \|x\|_\infty \le \|x\|_2 \le c_2 \|x\|_\infty$$

Answer:

$\forall x \in \mathbb{R}^n$ & $x \neq 0$

$$c_1 \le \frac{\|x\|_2}{\|x\|_\infty}$$

implies

$$c_1 = \inf_{x \neq 0} \frac{\|x\|_2}{\|x\|_\infty}$$

$$\Rightarrow c_1 = \inf_{\|x\|_\infty = 1} \|x\|_2$$

$$= \operatorname*{Min}_{\substack{\max |x_i| = 1 \\ 1 \le i \le n}} \left(\sqrt{\sum_{i=1}^{n} x_i^2} \right)$$

sup: supremum or least upper bound

inf: infimum or greatest lower bound

If $n=2$,

$$c_1 = \operatorname*{Min}_{\max\{|x_1|, |x_2|\} = 1} \left(\sqrt{x_1^2 + x_2^2} \right) = 1$$

$$\longrightarrow \begin{cases} |x_1| = 1 \\ |x_2| = 0 \end{cases} \text{ or } \begin{cases} |x_1| = 0 \\ |x_2| = 1 \end{cases}$$

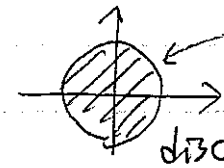
$$\Rightarrow \inf_{\|x\|_\infty = 1} \|x\|_2 = 1$$

4 Convex body

$$\|\cdot\| \longleftrightarrow B_{\|\cdot\|} = \{x \mid x \in \mathbb{R}^n,\ \|x\| < 1\}$$

Example:

1 $\|\cdot\|_2 \longleftrightarrow B_{\|\cdot\|_2} = \{x \mid x \in \mathbb{R}^n,\ \sqrt{\sum x_i^2} \le 1\}$



2 $\|\cdot\|_1 \longleftrightarrow B_{\|\cdot\|_1} \triangleq B_1 = \{x \mid x \in \mathbb{R}^n,\ \sum_{i=1}^{n} |x_i| \le$

If $n=2$, $|x_1| + |x_2| \le 1$

Redefine the norm with its convex body

$x, y \in B_{\|\cdot\|}$

$\alpha x + \beta y \in B_{\|\cdot\|}$ $\rightarrow \|\alpha x + \beta y\| \le 1$

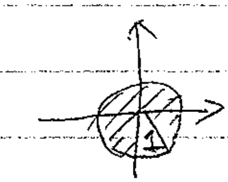
$\Rightarrow |\alpha|\,\|x\| + |\beta|\,\|y\| \le 1$

$\alpha, \beta \ge 0$

$\alpha + \beta = 1$

$$\|x\|_B \triangleq \inf\{\alpha \mid (\tfrac{1}{\alpha})x \in B_{\|\cdot\|}\}$$

Example: $B_2 = \{x \mid \|x\|_2 \le 1\}$



Example: Assume $x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\|x\|_B = \inf\{\alpha \mid \frac{1}{\alpha} x \in B_2$

$\alpha = ?$, i.e. find a smallest α s.t. $(\frac{1}{\alpha})\begin{pmatrix} 2 \\ 3 \end{pmatrix} \in B$

Sol: $\|\frac{1}{\alpha}\begin{pmatrix} 2 \\ 3 \end{pmatrix}\|_2 = 1 \Rightarrow \frac{1}{\alpha}\|\begin{pmatrix} 2 \\ 3 \end{pmatrix}\|_2 = 1 \Rightarrow \frac{1}{\alpha}\sqrt{13} = 1$

Example: Let $B = \{x \mid x \in \mathbb{R}^2,\ \frac{x^2}{16} + \frac{y^2}{25} \le 1\}$,

For $x = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$, Find $\|x\|_B = ?$ $\sqrt{\frac{25}{16} + \frac{36}{25}}$

Example: Let $B = \{x \mid x \in \mathbb{R}^3,\ \frac{x_1^2}{16} + \frac{x_2^2}{25} + \frac{x_3^2}{9} \le 1\}$

For $x = [5\ 6\ 7]^T$ Find $\|x\|_B = ?$