1.

Use the summation convention to write each of the following systems, state which indices are free and which are dummy indices, and fix the value of n:

$$(a)\quad \begin{aligned} c_{11}x_1 + c_{12}x_2 + c_{13}x_3 &= 2 \\ c_{21}x_1 + c_{22}x_2 + c_{23}x_3 &= -3 \\ c_{31}x_1 + c_{32}x_2 + c_{33}x_3 &= 5 \end{aligned} \qquad (b)\quad a_j^1x_1 + a_j^2x_2 + a_j^3x_3 + a_j^4x_4 = b_j \quad (j = 1, 2)$$

(*a*) Set $d_1 = 2$, $d_2 = -3$, and $d_3 = 5$. Then one can write the system as $c_{ij}x_j = d_i$ $(n = 3)$. The free index is i and the dummy index is j.

(*b*) Here, the range of the free index does not match that of the dummy index $(n = 4)$, and this fact must be indicated:

$$a_j^i x_i = b_j \qquad (j = 1, 2)$$

The free index is j and the dummy index is i.

2.

If $n = 2$, write out explicitly the triple summation $c_{rst}x^r y^s z^t$.

Any expansion technique that yields all $2^3 = 8$ terms will do. In this case we shall interpret the triplet *rst* as a three-digit integer, and list the terms in increasing order of that integer:

$$\begin{aligned} c_{rst}x^r y^s z^t = {} & c_{111}x^1y^1z^1 + c_{112}x^1y^1z^2 + c_{121}x^1y^2z^1 + c_{122}x^1y^2z^2 \\ & + c_{211}x^2y^1z^1 + c_{212}x^2y^1z^2 + c_{221}x^2y^2z^1 + c_{222}x^2y^2z^2 \end{aligned}$$

3.

Express $b^{ij}y_iy_j$ in terms of x-variables, if $y_i = c_{ij}x_j$ and $b^{ij}c_{ik} = \delta^j_k$.

$$b^{ij}y_iy_j = b^{ij}(c_{ir}x_r)(c_{js}x_s) = (b^{ij}c_{ir})x_rc_{js}x_s = \delta^j_r x_r c_{js}x_s = x_jc_{js}x_s = c_{ij}x_ix_j$$

4.

If $A_j = g_{jk}A^k$, show that $A^k = g^{jk}A_j$.

Multiply $A_j = g_{jk}A^k$ by g^{jq}.

Then $g^{jq}A_j = g^{jq}g_{jk}A^k = \delta^q_k A^k = A^q$, i.e. $A^q = g^{jq}A_j$ or $A^k = g^{jk}A_j$.

The tensors of rank one, A_j and A^k, are called *associated*. They represent the covariant a[...] variant components of a vector.

5.

EXAMPLE 2. Show that the sum of two vectors (defined in the familiar way) is a vector, but the scalar product is an invariant.

Solution: Let a_i and b_i denote the components of two vectors in a coordinate system x_i. Under a transformation $x_i^* = \alpha_{ij}x_j$, we know that

$$a_i^* = \alpha_{ij}a_j, \qquad b_i^* = \alpha_{ij}b_j$$

where a_i^* and b_i^* are the corresponding components of the two vectors in the new coordinate system x_i^*. Hence,

$$(a_i^* + b_i^*) = \alpha_{ij}a_j + \alpha_{ij}b_j = \alpha_{ij}(a_j + b_j)$$

so that the quantities $a_i + b_i$ are components of a vector, which is the sum of the two given vectors.

To show that the scalar product of two vectors is an invariant, we have to show that the quantity $a_i^*b_i^*$ remains numerically the same in any other coordinate system, that is, $a_i^*b_i^* = a_jb_j$. Now, according to the transformation law, we have

$$a_i^*b_i^* = (\alpha_{ip}a_p)(\alpha_{iq}b_q)$$

$$= \alpha_{ip}\alpha_{iq}a_pb_q$$

But $\alpha_{ip}\alpha_{iq} = \delta_{pq}$, hence,

$$a_i^*b_i^* = \delta_{pq}a_pb_q = a_pb_p$$

as is desired.