

## Tensor analysis HW2

1. Determine  $g_{ij}$  and  $g^{ij}$  in (a) cylindrical and (b) spherical coordinate.

Determine the metric tensor in (a) cylindrical and (b) spherical coordinates.

(a) As in Problem 7, Chapter 7,  $ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$ .

If  $x^1 = \rho$ ,  $x^2 = \phi$ ,  $x^3 = z$  then  $g_{11} = 1$ ,  $g_{22} = \rho^2$ ,  $g_{33} = 1$ ,  $g_{12} = g_{21} = 0$ ,  $g_{23} = g_{32} = 0$ ,  $g_{31} = g_{13} = 0$ .

In matrix form the metric tensor can be written 
$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) As in Problem 8(a), Chapter 7,  $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ .

If  $x^1 = r$ ,  $x^2 = \theta$ ,  $x^3 = \phi$  the metric tensor can be written 
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$
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In general for orthogonal coordinates,  $g_{jk} = 0$  for  $j \neq k$ . *z*

Determine the conjugate metric tensor in (a) cylindrical and (b) spherical coordinates.

(a) From Problem 30(a),  $g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho^2$

$$g^{11} = \frac{\text{cofactor of } g_{11}}{g} = \frac{1}{\rho^2} \begin{vmatrix} \rho^2 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$g^{22} = \frac{\text{cofactor of } g_{22}}{g} = \frac{1}{\rho^2} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{\rho^2}$$

$$g^{33} = \frac{\text{cofactor of } g_{33}}{g} = \frac{1}{\rho^2} \begin{vmatrix} 1 & 0 \\ 0 & \rho^2 \end{vmatrix} = 1$$

$$g^{12} = \frac{\text{cofactor of } g_{12}}{g} = -\frac{1}{\rho^2} \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

Similarly  $g^{jk} = 0$  if  $j \neq k$ . In matrix form the conjugate metric tensor can be represented by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{g^{ij} in cylindrical.}$$

(b) From Problem 30(b),  $g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{vmatrix} = r^4 \sin^2\theta$

As in part (a), we find  $g^{11} = 1$ ,  $g^{22} = \frac{1}{r^2}$ ,  $g^{33} = \frac{1}{r^2 \sin^2\theta}$  and  $g^{jk} = 0$  for  $j \neq k$ , and in matrix form this can be written

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2 \sin^2\theta \end{pmatrix}$$

2. Find (a)  $g$  and (b)  $g^{ij}$  corresponding to

$$ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6dx^1dx^2 + 4dx^2dx^3$$

Find (a)  $g$  and (b)  $g^{jk}$  corresponding to  $ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6dx^1dx^2 + 4dx^2dx^3$

(a)  $g_{11}=5, g_{22}=3, g_{33}=4, g_{12}=g_{21}=-3, g_{23}=g_{32}=2, g_{13}=g_{31}=0$ . Then  $g = \begin{vmatrix} 5 & -3 & 0 \\ -3 & 3 & 2 \\ 0 & 2 & 4 \end{vmatrix} = 4$ .

(b) The cofactors  $G(j,k)$  of  $g_{jk}$  are

$$G(1,1)=8, G(2,2)=20, G(3,3)=6, G(1,2)=G(2,1)=12, G(2,3)=G(3,2)=-10, G(1,3)=G(3,1)=-6$$

$$\text{Then } g^{11}=2, g^{22}=5, g^{33}=3/2, g^{12}=g^{21}=3, g^{23}=g^{32}=-5/2, g^{13}=g^{31}=-3/2$$

Note that the product of the matrices  $(g_{jk})$  and  $(g^{jk})$  is the unit matrix I, i.e.

$$\begin{pmatrix} 5 & -3 & 0 \\ -3 & 3 & 2 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 3 & -3/2 \\ 3 & 5 & -5/2 \\ -3/2 & -5/2 & 3/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. prove  $\frac{\partial g^{pq}}{\partial \xi^m} = -g^{pn}\Gamma_{mn}^q - g^{qn}\Gamma_{mn}^p$

(b)  $\frac{\partial}{\partial x^m} (g^{jk} g_{ij}) = \frac{\partial}{\partial x^m} (\delta_i^k) = 0$ . Then

$$g^{jk} \frac{\partial g_{ij}}{\partial x^m} + \frac{\partial g^{jk}}{\partial x^m} g_{ij} = 0 \quad \text{or} \quad g_{ij} \frac{\partial g^{jk}}{\partial x^m} = -g^{jk} \frac{\partial g_{ij}}{\partial x^m}$$

Multiplying by  $g^{ir}$ ,  $g^{ir} g_{ij} \frac{\partial g^{jk}}{\partial x^m} = -g^{ir} g^{jk} \frac{\partial g_{ij}}{\partial x^m}$

i.e.  $\delta_j^r \frac{\partial g^{jk}}{\partial x^m} = -g^{ir} g^{jk} ([im, j] + [jm, i])$

or  $\frac{\partial g^{rk}}{\partial x^m} = -g^{ir} \left\{ \begin{matrix} k \\ im \end{matrix} \right\} - g^{jk} \left\{ \begin{matrix} r \\ jm \end{matrix} \right\}$

and the result follows on replacing  $r, k, i, j$  by  $p, q, n, n$  respectively.

4. Determine  $\Gamma_{ij}^k$  in (a) rectangular, (b) cylindrical, and (c) spherical coordinates.

Determine the Christoffel symbols of the second kind in (a) rectangular, (b) cylindrical, and (c) spherical coordinates.

We can use the results of Problem 48, since for orthogonal coordinates  $g_{pq} = 0$  if  $p \neq q$ .

(a) In rectangular coordinates,  $g_{pp} = 1$  so that  $\left\{ \begin{smallmatrix} s \\ pq \end{smallmatrix} \right\} = 0$ .

(b) In cylindrical coordinates,  $x^1 = \rho$ ,  $x^2 = \phi$ ,  $x^3 = z$ , we have by Problem 30(a),  $g_{11} = 1$ ,  $g_{22} = \rho^2$ ,  $g_{33} = 1$ . The only non-zero Christoffel symbols of the second kind can occur where  $p = 2$ . These are

$$\left\{ \begin{smallmatrix} 1 \\ 22 \end{smallmatrix} \right\} = -\frac{1}{2g_{11}} \frac{\partial g_{22}}{\partial x^1} = -\frac{1}{2} \frac{\partial}{\partial \rho} (\rho^2) = -\rho$$

$$\left\{ \begin{smallmatrix} 2 \\ 21 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 2 \\ 12 \end{smallmatrix} \right\} = \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial x^1} = \frac{1}{2\rho^2} \frac{\partial}{\partial \rho} (\rho^2) = \frac{1}{\rho}$$

(c) In spherical coordinates,  $x^1 = r$ ,  $x^2 = \theta$ ,  $x^3 = \phi$ , we have by Prob. 30(b),  $g_{11} = 1$ ,  $g_{22} = r^2$ ,  $g_{33} = r^2 \sin^2 \theta$ . The only non-zero Christoffel symbols of the second kind can occur where  $p = 2$  or 3. These are

$$\left\{ \begin{smallmatrix} 1 \\ 22 \end{smallmatrix} \right\} = -\frac{1}{2g_{11}} \frac{\partial g_{22}}{\partial x^1} = -\frac{1}{2} \frac{\partial}{\partial r} (r^2) = -r$$

$$\left\{ \begin{smallmatrix} 2 \\ 21 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 2 \\ 12 \end{smallmatrix} \right\} = \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial x^1} = \frac{1}{2r^2} \frac{\partial}{\partial r} (r^2) = \frac{1}{r}$$

$$\left\{ \begin{smallmatrix} 1 \\ 33 \end{smallmatrix} \right\} = -\frac{1}{2g_{11}} \frac{\partial g_{33}}{\partial x^1} = -\frac{1}{2} \frac{\partial}{\partial r} (r^2 \sin^2 \theta) = -r \sin^2 \theta$$

$$\left\{ \begin{smallmatrix} 2 \\ 33 \end{smallmatrix} \right\} = -\frac{1}{2g_{22}} \frac{\partial g_{33}}{\partial x^2} = -\frac{1}{2r^2} \frac{\partial}{\partial \theta} (r^2 \sin^2 \theta) = -\sin \theta \cos \theta$$

$$\left\{ \begin{smallmatrix} 3 \\ 31 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 3 \\ 13 \end{smallmatrix} \right\} = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^1} = \frac{1}{2r^2 \sin^2 \theta} \frac{\partial}{\partial r} (r^2 \sin^2 \theta) = \frac{1}{r}$$

$$\left\{ \begin{smallmatrix} 3 \\ 32 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 3 \\ 23 \end{smallmatrix} \right\} = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^2} = \frac{1}{2r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} (r^2 \sin^2 \theta) = \cot \theta$$

又又 1 = 0  
 $\frac{\partial}{\partial \phi} = 0$   
 $\frac{\partial}{\partial \theta} = 0$   
 $\frac{\partial}{\partial r} = 0$

5. Prove that the covariant derivatives of (a)  $g_{ij}$ , (b)  $g^{ij}$ , (c)  $\delta_j^i$  are zero.

Prove that the covariant derivatives of (a)  $g_{jk}$ , (b)  $g^{jk}$ , (c)  $\delta_k^j$  are zero.

$$\begin{aligned} (a) \quad g_{jk,q} &= \frac{\partial g_{jk}}{\partial x^q} - \left\{ \begin{smallmatrix} s \\ jq \end{smallmatrix} \right\} g_{sk} - \left\{ \begin{smallmatrix} s \\ kq \end{smallmatrix} \right\} g_{js} \\ &= \frac{\partial g_{jk}}{\partial x^q} - [jq, k] - [kq, j] = 0 \quad \text{by Problem 45(a).} \end{aligned}$$

$$(b) \quad g^{jk,q} = \frac{\partial g^{jk}}{\partial x^q} + \left\{ \begin{smallmatrix} j \\ qs \end{smallmatrix} \right\} g^{sk} + \left\{ \begin{smallmatrix} k \\ qs \end{smallmatrix} \right\} g^{js} = 0 \quad \text{by Problem 45(b).}$$

$$(c) \quad \delta_k^j{}_{,q} = \frac{\partial \delta_k^j}{\partial x^q} - \left\{ \begin{smallmatrix} s \\ kq \end{smallmatrix} \right\} \delta_s^j + \left\{ \begin{smallmatrix} j \\ qs \end{smallmatrix} \right\} \delta_k^s = 0 - \left\{ \begin{smallmatrix} j \\ kq \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} j \\ qk \end{smallmatrix} \right\} = 0$$