1.

$$G=\begin{bmatrix}1&0&0\\0&(x^1)^2&0\\0&0&(x^1\sin x^2)^2\end{bmatrix},\quad g=(x^1)^4\sin^2 x^2$$

$$\nabla^2 f=\mathrm{div}\left(g^{ij}\frac{\partial f}{\partial x^j}\right)=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\,g^{ij}\frac{\partial f}{\partial x^j}\right)$$

$$\sqrt{g}\,g^{ij}\frac{\partial f}{\partial x^j}=(x^1)^2(\sin x^2)\left(g^{i1}\frac{\partial f}{\partial x^1}+g^{i2}\frac{\partial f}{\partial x^2}+g^{i3}\frac{\partial f}{\partial x^3}\right)$$

$$\Rightarrow \sqrt{g}\,g^{1j}\frac{\partial f}{\partial x^j}=(x^1)^2(\sin x^2)\frac{\partial f}{\partial x^1}$$

$$\sqrt{g}\,g^{2j}\frac{\partial f}{\partial x^j}=(x^1)^2(\sin x^2)\frac{1}{(x^1)^2}\frac{\partial f}{\partial x^2}=(\sin x^2)\frac{\partial f}{\partial x^2}$$

$$\sqrt{g}\,g^{3j}\frac{\partial f}{\partial x^j}=(x^1)^2(\sin x^2)\frac{1}{(x^1\sin x^2)^2}\frac{\partial f}{\partial x^3}=(\csc x^2)\frac{\partial f}{\partial x^3}$$

$$\therefore \frac{\partial}{\partial x^i}\left[\sqrt{g}\,g^{ij}\frac{\partial f}{\partial x^j}\right]=\frac{\partial}{\partial x^1}\left[(x^1)^2(\sin x^2)\frac{\partial f}{\partial x^1}\right]+\frac{\partial}{\partial x^2}\left[(\sin x^2)\frac{\partial f}{\partial x^2}\right]+\frac{\partial}{\partial x^3}\left[(\csc x^2)\frac{\partial f}{\partial x^3}\right]$$

$$=2x^1(\sin x^2)\frac{\partial f}{\partial x^1}+(x^1)^2(\sin x^2)\frac{\partial^2 f}{(\partial x^1)^2}+(\cos x^2)\frac{\partial f}{\partial x^2}+(\sin x^2)\frac{\partial^2 f}{(\partial x^2)^2}$$
$$+(\csc x^2)\frac{\partial^2 f}{(\partial x^3)^2}$$

$r=x^1,\ \theta=x^2,\ \phi=x^3$ 代入

$$\nabla^2 f=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left[\sqrt{g}\,g^{ij}\frac{\partial f}{\partial x^j}\right]$$

$$=\frac{1}{r^2\sin\theta}\left[(2r\sin\theta)\frac{\partial f}{\partial r}+(r^2\sin\theta)\frac{\partial^2 f}{\partial r^2}+(\cos\theta)\frac{\partial f}{\partial\theta}+\right.$$
$$\left.(\sin\theta)\frac{\partial^2 f}{\partial\theta^2}+(\csc\theta)\frac{\partial^2 f}{\partial\phi^2}\right]$$

$$=\frac{\partial^2 f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}+\frac{2}{r}\frac{\partial f}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial f}{\partial\theta}$$

2.

$$\mathrm{div}\,\mathbf{u}=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}\,u^i)=\frac{\partial u^i}{\partial x^i}+u^i\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g})$$

$$=\frac{\partial u^i}{\partial x^i}+u^i\frac{1}{(x^1)^2\sin x^2}\frac{\partial}{\partial x^i}\left[(x^1)^2\sin x^2\right]$$

$$=\frac{\partial u^i}{\partial x^i}+u^1\left(\frac{2}{x^1}\right)+u^2\left(\frac{\cos x^2}{\sin x^2}\right)$$

$$\therefore\ \mathrm{div}\,\mathbf{u}=\frac{\partial u^1}{\partial\rho}+\frac{\partial u^2}{\partial\phi}+\frac{\partial u^3}{\partial\theta}+\frac{2}{\rho}u^1+(\cot\phi)u^2$$

3. (a) $\nabla V = \frac{\partial}{\partial \xi^j}(V^i \varepsilon_i)\varepsilon_j = V^i_{,j}\varepsilon_i \varepsilon_j$

$$V^i_{,j} = \frac{\partial V^i}{\partial \xi^j} - V^k \Gamma^i_{kj}$$

$$\Gamma^1_{22} = -r,\ \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{r},\ \Gamma^1_{33} = -r\sin\theta,\ \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{31} = \Gamma^3_{13} = \frac{1}{r},\ \Gamma^3_{32} = \Gamma^3_{23} = \cot\theta$$

$$V^1_{,1} = \frac{\partial V_1}{\partial r}$$

$$V^1_{,2} = \frac{\partial V^1}{\partial \theta} - V^k\Gamma^1_{k2} = \frac{\partial V^1}{\partial \theta} - V^2\Gamma^1_{22} = \frac{\partial V^1}{\partial \theta} + V^2 r$$

$$V^1_{,3} = \frac{\partial V^1}{\partial \phi} - V^k\Gamma^1_{k3} = \frac{\partial V^1}{\partial \phi} - V^3\Gamma^1_{33} = \frac{\partial V^1}{\partial \phi} + V^3(r\sin\theta)$$

$$V^2_{,1} = \frac{\partial V^2}{\partial r} - V^k\Gamma^2_{k1} = \frac{\partial V^2}{\partial r} - V^2\Gamma^2_{21} = \frac{\partial V^2}{\partial r} - V^2\cdot\frac{1}{r}$$

$$V^2_{,2} = \frac{\partial V^2}{\partial \theta} - V^1\Gamma^2_{12} = \frac{\partial V^2}{\partial \theta} - V^1\left(\frac{1}{r}\right)$$

$$V^2_{,3} = \frac{\partial V^2}{\partial \phi} - V^k\Gamma^2_{k3} = \frac{\partial V^2}{\partial \phi} - V^3\Gamma^2_{33} = \frac{\partial V^2}{\partial \phi} + V^3(\sin\theta\cos\theta)$$

$$V^3_{,1} = \frac{\partial V^3}{\partial r} - V^k\Gamma^3_{k1} = \frac{\partial V^3}{\partial r} - V^3\Gamma^3_{31} = \frac{\partial V^3}{\partial r} - V^3\cdot\frac{1}{r}$$

$$V^3_{,2} = \frac{\partial V^3}{\partial \theta} - V^k\Gamma^3_{k2} = \frac{\partial V^3}{\partial \theta} - V^3\Gamma^3_{32} = \frac{\partial V^3}{\partial \theta} - V^3\cot\theta$$

$$V^3_{,3} = \frac{\partial V^3}{\partial \phi} - V^k\Gamma^3_{k3} = \frac{\partial V^3}{\partial \phi} - V^1\Gamma^3_{13} - V^2\Gamma^3_{23} = \frac{\partial V^3}{\partial \phi} - V^1\frac{1}{r} - V^2\cot\theta$$

$$\nabla V = V^1_{,1}\varepsilon_1\varepsilon_1 + V^1_{,2}\varepsilon_1\varepsilon_2 + V^1_{,3}\varepsilon_1\varepsilon_3 + V^2_{,1}\varepsilon_1\varepsilon_2 + V^2_{,2}\varepsilon_2\varepsilon_2 + V^2_{,3}\varepsilon_2\varepsilon_3$$
$$+ V^3_{,1}\varepsilon_3\varepsilon_1 + V^3_{,2}\varepsilon_3\varepsilon_2 + V^3_{,3}\varepsilon_3\varepsilon_3$$

(b) $V\nabla V = V^k\varepsilon_k V^i_{,j}\varepsilon_i\varepsilon_j$

$$V\cdot\nabla V = (V^k\varepsilon_k)\cdot(V^i_{,j}\varepsilon_i\varepsilon_j) = V^kV^i_{,j}\,\varepsilon_k\cdot\varepsilon_i\varepsilon_j = V^kV^i_{,j}\,g_{ki}\varepsilon_j$$

for spherical coordinate.

$$V\cdot\nabla V = V^1V^1_{,j}\,g_{11}\varepsilon_j + V^2V^2_{,j}\,g_{22}\varepsilon_j + V^3V^3_{,j}\,g_{22}\varepsilon_j$$

$$= V^1 V^1_{,1} g_{11} \varepsilon_1 + V^1 V^1_{,2} g_{11} \varepsilon_2 + V^1 V^1_{,3} g_{11} \varepsilon_3 + V^2 V^2_{,1} g_{22} \varepsilon_1 + V^2 V^2_{,2} g_{22} \varepsilon_2 + V^2 V^2_{,3} g_{22} \varepsilon_3$$
$$+ V^3 V^3_{,1} g_{33} \varepsilon_1 + V^3 V^3_{,2} g_{33} \varepsilon_2 + V^3 V^3_{,3} g_{33} \varepsilon_3$$

$$= \varepsilon_1 \left(V^1 V^1_{,1} g_{11} + V^2 V^2_{,1} g_{22} + V^3 V^3_{,1} g_{33} \right) +$$
$$\varepsilon_2 \left(V^1 V^1_{,2} g_{11} + V^2 V^2_{,2} g_{22} + V^3 V^3_{,2} g_{33} \right) +$$
$$\varepsilon_3 \left(V^1 V^1_{,3} g_{11} + V^2 V^2_{,3} g_{22} + V^3 V^3_{,3} g_{33} \right)$$

$g_{11} = 1$, $g_{22} = r^2$, $g_{33} = r^2 \sin^2\theta$

$V^i_{,j}$ 值參考(a)小題

(c)

$$\nabla\nabla V = \frac{\partial}{\partial \xi^k} \varepsilon_k V^i_{,j} \varepsilon_i \varepsilon_j$$

$$\nabla \cdot \nabla V = \frac{\partial}{\partial \xi^k} V^i_{,j} \varepsilon_k \cdot \varepsilon_i \varepsilon_j = \frac{\partial}{\partial \xi^k} V^i_{,j} g_{ki} \varepsilon_j$$

for spherical coordinate

$$\nabla \cdot \nabla V = \frac{\partial}{\partial \xi^i} V^i_{,j} g_{ii} \varepsilon_j = \frac{\partial}{\partial \xi^1} V^1_{,j} g_{11} \varepsilon_j + \frac{\partial}{\partial \xi^2} V^2_{,j} g_{22} \varepsilon_j + \frac{\partial}{\partial \xi^3} V^3_{,j} g_{33} \varepsilon_j$$

$$= \frac{\partial}{\partial \xi^1} V^1_{,1} g_{11} \varepsilon_1 + \frac{\partial}{\partial \xi^1} V^1_{,2} g_{11} \varepsilon_2 + \frac{\partial}{\partial \xi^1} V^1_{,3} g_{11} \varepsilon_3 + \frac{\partial}{\partial \xi^2} V^2_{,1} g_{22} \varepsilon_1 + \frac{\partial}{\partial \xi^2} V^2_{,2} g_{22} \varepsilon_2 + \frac{\partial}{\partial \xi^2} V^2_{,3} g_{22} \varepsilon_3$$
$$+ \frac{\partial}{\partial \xi^3} V^3_{,1} g_{33} \varepsilon_1 + \frac{\partial}{\partial \xi^3} V^3_{,2} g_{33} \varepsilon_2 + \frac{\partial}{\partial \xi^3} V^3_{,3} g_{33} \varepsilon_3$$

$$= \varepsilon_1 \left(\frac{\partial}{\partial \xi^1} V^1_{,1} g_{11} + \frac{\partial}{\partial \xi^2} V^2_{,1} g_{22} + \frac{\partial}{\partial \xi^3} V^3_{,1} g_{33} \right) +$$
$$\varepsilon_2 \left(\frac{\partial}{\partial \xi^1} V^1_{,2} g_{11} + \frac{\partial}{\partial \xi^2} V^2_{,2} g_{22} + \frac{\partial}{\partial \xi^3} V^3_{,2} g_{33} \right) +$$
$$\varepsilon_3 \left(\frac{\partial}{\partial \xi^1} V^1_{,3} g_{11} + \frac{\partial}{\partial \xi^2} V^2_{,3} g_{22} + \frac{\partial}{\partial \xi^3} V^3_{,3} g_{33} \right)$$

g_{ij}, $V^i_{,j}$ 值參(a)、(b)小題

4.

(a) $R_{ij} = R^k_{ijk} = R^1_{ij1} + R^2_{ij2} + R^3_{ij3}$, $g_{ij} = 0$, for $i \neq j$

$R_{ij} = g^{11} R_{1ij1} + g^{22} R_{2ij2} + g^{33} R_{3ij3}$

$R_{1221} = -R_{1212} = -\frac{1}{2x^1}$

$R_{2332} = -R_{2323} = -\frac{1}{2x^2}$

$R_{3123} = -R_{3132} = -\frac{1}{2x^1}$

$R_{2112} = -\frac{1}{2x^1}$

$R_{3223} = -\frac{1}{2x^2}$

$R_{3213} = -\frac{1}{2x^1}$

$R_{11} = g^{22} R_{2112} = -\frac{1}{4(x^1)^2}$

$R_{22} = g^{11} R_{1221} + g^{33} R_{3223} = -\frac{1}{2x^1} - \frac{1}{4(x^2)^2}$

$R_{33} = g^{22} R_{2332} = -\frac{1}{4x^1x^2}$

$R_{12} = g^{33} R_{3123} = -\frac{1}{4x^1x^2} = g^{33} R_{3213} = R_{21}$

(b) $R^i_j = g^{ik} R_{kj} = g^{ii} R_{ij}$ (not summation on i)

$R^1_1 = g_{11} R_{11} = -\frac{1}{4(x^1)^2}$

$R^2_2 = g_{22} R_{22} = -\frac{1}{4(x^1)^2} - \frac{1}{8x^1(x^2)^2}$

$R^3_3 = g_{33} R_{33} = -\frac{1}{8x^1(x^2)^2}$

(c)

$$R = R^1_1 + R^2_2 + R^3_3 = (1)\left[-\frac{1}{4(x^1)^2}\right] + \left(\frac{1}{2x^1}\right)\left[-\frac{1}{2x^1} - \frac{1}{4(x^2)^2}\right] + \left(\frac{1}{2x^2}\right)\left(-\frac{1}{4x^1x^2}\right)$$

$$= -\frac{x^1 + 2(x^2)^2}{(2x^1x^2)^2}$$

5.

$$f_{k,ij} = (f_{k,i})_{,j} = \frac{\partial}{\partial \xi^j}(f_{k,i}) - \Gamma^r_{kj}(f_{r,i}) - \Gamma^r_{ki}(f_{j,r})$$

$$f_{k,i} = \frac{\partial f_k}{\partial \xi^i} - \Gamma^p_{ki} f_p$$ 代入

$$\Rightarrow f_{k,ij} = \frac{\partial^2 f_k}{\partial \xi^j \partial \xi^i} - \frac{\partial \Gamma^p_{ki}}{\partial \xi^j} f_p - \Gamma_{ki} \frac{\partial f_p}{\partial \xi^j} - \Gamma^r_{kj} \frac{\partial f_r}{\partial \xi^k} + \Gamma^r_{kj}\Gamma^p_{ri} f_p$$
$$- \Gamma^r_{ij} \frac{\partial f_k}{\partial \xi^r} + \Gamma^r_{ij}\Gamma^p_{kr} f_p \quad -①$$

i, j interchange.

$$f_{k,ji} = \frac{\partial^2 f_k}{\partial \xi^i \xi^j} - \frac{\partial \Gamma^p_{kj}}{\partial \xi^i} f_p - \Gamma^p_{kj} \frac{\partial f_p}{\partial \xi^i} - \Gamma^r_{ki} \frac{\partial f_r}{\partial \xi^j} + \Gamma^r_{ki}\Gamma^p_{rj} f_p$$
$$- \Gamma^r_{ji} \frac{\partial f_k}{\partial \xi^r} + \Gamma^r_{ji}\Gamma^p_{kr} f_p \quad -②$$

① − ②.

$$f_{k,ij} - f_{k,ji} = - \frac{\partial \Gamma^p_{ki}}{\partial \xi^j} f_p + \Gamma^r_{kj}\Gamma^p_{ri} f_p + \frac{\partial \Gamma^p_{kj}}{\partial \xi^i} f_p - \Gamma^r_{ki}\Gamma^p_{rj} f_p$$
$$= \left(\frac{\partial \Gamma^p_{ki}}{\partial \xi^j} - \frac{\partial \Gamma^p_{kj}}{\partial \xi^i} + \Gamma^r_{kj}\Gamma^p_{ri} - \Gamma^r_{ki}\Gamma^p_{rj} \right) f_p$$

6.

(a) $\Gamma^1_{22} = -x^1$, $\Gamma^1_{33} = -x^1\sin^2 x^2$

$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{x^1}$, $\Gamma^2_{33} = -\sin x^2 \cos x^2$

$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{x^1}$, $\Gamma^3_{23} = \Gamma^3_{32} = \cot x^2$

$$R^1_{212} = \frac{\partial \Gamma^1_{22}}{\partial x^1} - \frac{\partial \Gamma^1_{21}}{\partial x^2} + \Gamma^r_{22}\Gamma^1_{r1} - \Gamma^r_{21}\Gamma^1_{r2} = -1 - \Gamma^1_{22}\Gamma^1_{11} - \Gamma^2_{21}\Gamma^1_{22} = 0$$

$$R^1_{313} = \frac{\partial \Gamma^1_{33}}{\partial x^1} - \frac{\partial \Gamma^1_{31}}{\partial x^1} + \Gamma^r_{33}\Gamma^1_{r1} - \Gamma^r_{31}\Gamma^1_{r3} = -\sin^2 x^2 + \Gamma^1_{33}\Gamma^1_{11} + \Gamma^3_{31}\Gamma^1_{33} = 0$$

$$R^2_{323} = \frac{\partial \Gamma^2_{33}}{\partial x^2} - \frac{\partial \Gamma^2_{32}}{\partial x^3} + \Gamma^r_{33}\Gamma^2_{r2} - \Gamma^r_{32}\Gamma^2_{r3} = -\cos 2x^2 - \sin^2 x^2 + \cos^2 x^2 = 0.$$

$$R^1_{213} = \frac{\partial \Gamma^1_{23}}{\partial x^1} - \frac{\partial \Gamma^1_{21}}{\partial x^3} + \Gamma^r_{23}\Gamma^1_{r1} - \Gamma^r_{21}\Gamma^1_{r3} = \Gamma^3_{23}\Gamma^1_{31} - \Gamma^2_{21}\Gamma^1_{23} = 0$$

$$R^1_{232} = \frac{\partial \Gamma^1_{22}}{\partial x^3} - \frac{\partial \Gamma^1_{23}}{\partial x^2} + \Gamma^r_{22}\Gamma^1_{r3} - \Gamma^r_{23}\Gamma^1_{r2} = \Gamma^1_{22}\Gamma^1_{13} - \Gamma^3_{23}\Gamma^1_{32} = 0$$

$$R^1_{323} = \frac{\partial \Gamma^1_{33}}{\partial x^2} - \frac{\partial \Gamma^1_{32}}{\partial x^3} + \Gamma^r_{33}\Gamma^1_{r2} - \Gamma^r_{22}\Gamma^1_{r3} = -2x^1\sin x^2\cos x^2 + \Gamma^2_{33}\Gamma^1_{22} - \Gamma^3_{32}\Gamma^1_{33}$$

$$= -2x^1\sin x^2\cos x^2 + x^1\sin x^2\cos x^2 + (\cot x^2)(x^1\sin^2 x^2) = 0$$

$\therefore R_{ijkl} = 0$, for all i, j, k, l

(b) $\Gamma^1_{22} = -x^1$, $\Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{x^1}$

$$R^1_{212} = \frac{\partial \Gamma^1_{22}}{\partial x^1} - \frac{\partial \Gamma^1_{21}}{\partial x^2} + \Gamma^r_{22}\Gamma^1_{r1} - \Gamma^r_{21}\Gamma^1_{r2} = 0$$

$R^1_{313} = 0$

$R^2_{323} = 0$

$R^1_{213} = 0$

$R^1_{232} = R^2_{123} = 0$

$R^1_{323} = R^3_{132} = 0$

Note:習題第 5 題，講義上推導有錯誤，請以習題解答為主