3. (a) $\nabla V = \frac{\partial}{\partial \xi^j}(V^i \varepsilon_i)\varepsilon^j = V^i_{,j}\,\varepsilon_i \varepsilon^j$

$V^i_{,j} = \frac{\partial V^i}{\partial \xi^j} + V^k \Gamma^i_{kj}$

$\Gamma^1_{22} = -r,\ \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{r},\ \Gamma^1_{33} = -r\sin\theta,\ \Gamma^2_{33} = -\sin\theta\cos\theta$

$\Gamma^3_{31} = \Gamma^3_{13} = \frac{1}{r},\ \Gamma^3_{32} = \Gamma^3_{23} = \cot\theta$

$V^1_{,1} = \frac{\partial V_1}{\partial r}$

$V^1_{,2} = \frac{\partial V^1}{\partial \theta} + V^k \Gamma^1_{k2} = \frac{\partial V^1}{\partial \theta} + V^2 \Gamma^1_{22} = \frac{\partial V^1}{\partial \theta} - V^2 r$

$V^1_{,3} = \frac{\partial V^1}{\partial \phi} + V^k \Gamma^1_{k3} = \frac{\partial V^1}{\partial \phi} + V^3 \Gamma^1_{33} = \frac{\partial V^1}{\partial \phi} - V^3 (r\sin\theta)$

$V^2_{,1} = \frac{\partial V^2}{\partial r} + V^k \Gamma^2_{k1} = \frac{\partial V^2}{\partial r} + V^2 \Gamma^2_{21} = \frac{\partial V^2}{\partial r} + V^2 \cdot \frac{1}{r}$

$V^2_{,2} = \frac{\partial V^2}{\partial \theta} + V^1 \Gamma^2_{12} = \frac{\partial V^2}{\partial \theta} + V^1 \left(\frac{1}{r}\right)$

$V^2_{,3} = \frac{\partial V^2}{\partial \phi} + V^k \Gamma^2_{k3} = \frac{\partial V^2}{\partial \phi} + V^3 \Gamma^2_{33} = \frac{\partial V^2}{\partial \phi} - V^3 (\sin\theta\cos\theta)$

$V^3_{,1} = \frac{\partial V^3}{\partial r} + V^k \Gamma^3_{k1} = \frac{\partial V^3}{\partial r} + V^3 \Gamma^3_{31} = \frac{\partial V^3}{\partial r} + V^3 \cdot \frac{1}{r}$

$V^3_{,2} = \frac{\partial V^3}{\partial \theta} + V^k \Gamma^3_{k2} = \frac{\partial V^3}{\partial \theta} + V^3 \Gamma^3_{32} = \frac{\partial V^3}{\partial \theta} + V^3 \cot\theta$

$V^3_{,3} = \frac{\partial V^3}{\partial \phi} + V^k \Gamma^3_{k3} = \frac{\partial V^3}{\partial \phi} + V^1 \Gamma^3_{13} + V^2 \Gamma^3_{23} = \frac{\partial V^3}{\partial \phi} + V^1 \frac{1}{r} + V^2 \cot\theta$

$\nabla V = V^1_{,1}\varepsilon_1\varepsilon^1 + V^1_{,2}\varepsilon_1\varepsilon^2 + V^1_{,3}\varepsilon_1\varepsilon^3 + V^2_{,1}\varepsilon_2\varepsilon^1 + V^2_{,2}\varepsilon_2\varepsilon^2 + V^2_{,3}\varepsilon_2\varepsilon^3$
$+ V^3_{,1}\varepsilon_3\varepsilon^1 + V^3_{,2}\varepsilon_3\varepsilon^2 + V^3_{,3}\varepsilon_3\varepsilon^3$

(b) $V\nabla V = V^k \varepsilon_k V^i_{,j}\,\varepsilon_i \varepsilon^j$

$V\cdot\nabla V = (V^k\varepsilon_k)\cdot(V^i_{,j}\,\varepsilon_i\varepsilon^j) = V^k V^i_{,j}\,\varepsilon_k\cdot\varepsilon_i\varepsilon^j = V^k V^i_{,j}\,g_{ki}\varepsilon^j$

for spherical coordinate.

$V\cdot\nabla V = V^1 V^1_{,j}\,g_{11}\varepsilon^j + V^2 V^2_{,j}\,g_{22}\varepsilon^j + V^3 V^3_{,j}\,g_{22}\varepsilon^j$

$$= V^1 V^1_{,1} g_{11} \varepsilon^1 + V^1 V^1_{,2} g_{11} \varepsilon^2 + V^1 V^1_{,3} g_{11} \varepsilon^3 + V^2 V^2_{,1} g_{22} \varepsilon^1 + V^2 V^2_{,2} g_{22} \varepsilon^2 + V^2 V^2_{,3} g_{22} \varepsilon^3$$
$$+ V^3 V^3_{,1} g_{33} \varepsilon^1 + V^3 V^3_{,2} g_{33} \varepsilon^2 + V^3 V^3_{,3} g_{33} \varepsilon^3$$

$$= \varepsilon^1 (V^1 V^1_{,1} g_{11} + V^2 V^2_{,1} g_{22} + V^3 V^3_{,1} g_{33}) +$$
$$\varepsilon^2 (V^1 V^1_{,2} g_{11} + V^2 V^2_{,2} g_{22} + V^3 V^3_{,2} g_{33}) +$$
$$\varepsilon^3 (V^1 V^1_{,3} g_{11} + V^2 V^2_{,3} g_{22} + V^3 V^3_{,3} g_{33})$$

$g_{11} = 1$, $g_{22} = r^2$, $g_{33} = r^2 \sin^2\theta$

$V^i_{,j}$ 值參考(a)小題

(c) $\nabla\nabla V = \frac{\partial}{\partial \xi^k} \varepsilon^k V^i_{,j} \varepsilon_i \varepsilon^j$

$\nabla \cdot \nabla V = \frac{\partial}{\partial \xi^k} V^i_{,j} \delta^k_i \varepsilon^j = \frac{\partial}{\partial \xi^i} V^i_{,j} \varepsilon^j$

in spherical coordinate

$$\nabla \cdot \nabla V = \frac{\partial}{\partial \xi^1} V^1_{,j} \varepsilon^j + \frac{\partial}{\partial \xi^2} V^2_{,j} \varepsilon^j + \frac{\partial}{\partial \xi^3} V^3_{,j} \varepsilon^j$$
$$= \frac{\partial}{\partial \xi^1} V^1_{,1} \varepsilon^1 + \frac{\partial}{\partial \xi^1} V^1_{,2} \varepsilon^2 + \frac{\partial}{\partial \xi^1} V^1_{,3} \varepsilon^3 + \frac{\partial}{\partial \xi^2} V^2_{,1} \varepsilon^1 + \frac{\partial}{\partial \xi^2} V^2_{,2} \varepsilon^2 + \frac{\partial}{\partial \xi^2} V^2_{,3} \varepsilon^3$$
$$+ \frac{\partial}{\partial \xi^3} V^3_{,1} \varepsilon^1 + \frac{\partial}{\partial \xi^3} V^3_{,2} \varepsilon^2 + \frac{\partial}{\partial \xi^3} V^3_{,3} \varepsilon^3$$
$$= \varepsilon^1 (\frac{\partial}{\partial \xi^1} V^1_{,1} + \frac{\partial}{\partial \xi^2} V^2_{,1} + \frac{\partial}{\partial \xi^3} V^3_{,1}) +$$
$$\varepsilon^2 (\frac{\partial}{\partial \xi^1} V^1_{,2} + \frac{\partial}{\partial \xi^2} V^2_{,2} + \frac{\partial}{\partial \xi^3} V^3_{,2}) +$$
$$\varepsilon^3 (\frac{\partial}{\partial \xi^1} V^1_{,3} + \frac{\partial}{\partial \xi^2} V^2_{,3} + \frac{\partial}{\partial \xi^3} V^3_{,3}).$$

$V^i_{,j}$ 值參考(a)小題