Chapter I Eigenfunction Expansion

I-1. Introduction:

Consider the eq.:

$$y'' + d_1(x)y' + d_2(x)y = 0, \quad a < x < b$$

with the b.c's:

$$y(a) = y(b) = 0$$

The g.s. is $y(x) = c_1u_1(x) + c_2u_2(x)$ where u_1,u_2 are linearly indep. fucs. and c_1,c_2 are arb. consts.

For b.c's:

$$\Rightarrow \quad c_1u_1(a) + c_2u_2(a) = 0$$

$$c_1u_1(b) + c_2u_2(b) = 0$$

The condition of nontrivial sol. of c_1,c_2 to be existed if $\Rightarrow$

$$\begin{vmatrix} u_1(a) & u_2(a) \\ u_2(b) & u_2(b) \end{vmatrix} = 0 \quad \text{(char. eq.)}$$

In many cases, one, or both d_1(x), d_2(x) and hence the sols u_1(x), u_2(x) depend upon a parameter λ which may take on various constant value. In such cases, the char. eq. may be satisfied for certain value of λ, say $\lambda = \lambda_1,\lambda_2,...$, and for each λ, a sol. is then obtained. Prob. of this sort are known as char-value problem (or eigenvalue prob). The value of λ for which nontrivial sols. exist are called the char.-value (or eigenvalue). And the corresponding sols are called the char. functions (or eigenfunctions)

Ex 1:The Euler Column

$$\frac{d^2v}{dx^2} + \lambda^2 v = 0 \quad where \quad \lambda^2 = \frac{P}{EI}$$

The g.s.

$$v(x) = c_1 \sin \lambda x + c_2 \cos \lambda x$$

For the b.c's: $v(0) = v(l) = 0 \Rightarrow \quad c_2 = 0$

and $c_1 \sin \lambda l = 0$ for nontrivial solution $c_1 \neq 0$

$\Rightarrow \sin \lambda l = 0$

$i.e. \lambda l = n\pi, n = 1, 2, ...$

so that we obtain the eigenvalues $\lambda_n = \frac{n\pi}{l}$, $n = 1, 2, ...$ and the corresponding eigenfucs (nontrivial sols) are $v_n(x) = sin \lambda_n x = \phi_n(x)$ say.

According the analysis, we will have $v(x) \equiv 0$ unless the end force P such that

$\lambda_n^2 = \frac{P_n}{EI} = \frac{n^2\pi^2}{l^2}$

in which the delection is given.

Physically, $P_n = \frac{n^2\pi^2 EI}{l^2}$ are called the "buckling loads" and $\phi_n(x) = \sin \frac{n\pi x}{l}$ are the "buckling modes".

For design purposes, the lowest buckling load $P_1 = \frac{\pi^2 EI}{l^2}$ is of most interesrt and is called the "critical buckling load", (Euler load)

Ex2: Temperature distribution in a slab

Eq:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad (0 < x < l, t > 0)$$

$$I.C.: \quad u(x, 0) = f(x), \quad 0 \leq x \leq l$$

$$B.C: \quad u(0, t) = 0, \quad u(l, t) = 0, \quad t \geq 0$$

By the separation of variables, assume

u(x,t)= X(x)T(t)

Eq. $\Rightarrow$

$$XT' = \alpha X''T$$

$$\div \alpha XT \Rightarrow \frac{T'}{\alpha T} = \frac{X''}{X} = k \quad (const)$$

$\Rightarrow$ the separated eqs.

$$X'' - kX = 0 \quad \text{with} \quad X(0) = X(l) = 0$$

$$T' - \alpha kT = 0$$

Sol. of $X(x)$:

(i) If $k > 0$, let $k = \lambda^2 \quad (\lambda : real)$

$\Rightarrow X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

For b.c's $\Rightarrow$

$$c_1 + c_2 = 0$$
$$c_1 e^{\lambda l} + c_2 e^{-\lambda l} = 0$$

since

$$\begin{vmatrix} 1 & 1 \\ e^{\lambda l} & e^{-\lambda l} \end{vmatrix} \neq 0$$

$\Rightarrow$ only trivial sol. of c_1, c_2 is obtained.

Hence this is not the case!

(ii) If k=0 $\Rightarrow X(x) = c_1 + c_2 x$

For b.c's$\Rightarrow$ $c_1 = 0 \quad , \quad c_2 l = 0$

$\Rightarrow$ only trivial sol. is obtained.

(iii) If k$<$ 0, let k=-λ^2 (λ:real)

$\Rightarrow \quad X'' + \lambda^2 X = 0$

$X(0) = X(l) = 0$

From previous example, we have the nontrivial sols., the eigenfucs are

$$\lambda_n = \frac{n\pi}{l}, \quad X_n(x) = \sin \lambda_n x$$

The determination of the separation const. may also be obtained from the physical consideration.

(a) Spacial dependence : From $X(0) = X(l) = 0$, it behaves some kind of periodic phenomenon $\Rightarrow$ the exponential fucs. may not exists, and the trigonometric fucs. may exist $\Rightarrow$ k is negative.

(b) Time dependence : The heat conduction is itself a diffusion behavior, and it will present a exponential decay character in time history $\Rightarrow$ k is negative.

Solution of T(t)

$$T_n' + \alpha {\lambda_n}^2 T_n = 0$$

$$\Rightarrow T_n(t) = c_n e^{-\alpha {\lambda_n}^2 t}$$

Hence we have the g.s

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} e^{-\alpha(\frac{n\pi}{l})^2 t}$$

For the I.C. $\Rightarrow$ u(x,0)=f(x)=$\sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l}$

It is to expand a fucs. f(x) in terms of the eigenfucs $X_n(x)$ and it likes to expand a vector in a linear vector space.

Now question arise:

(a) Is the set $X_n(x) = \sin \frac{n\pi x}{l}, n = 1, 2, ...$ complete in $0 \leq x \leq l$?

(b) How do we compute the coeffs. C_n?

I-2. Linear Vector Space:

A system of n real numbers $x_1, x_2, ..., x_n$ is called an n-dim vector or a vector in n-dim space, and denoted by the letter $\underset{\sim}{x}$, the number $x_i (i = 1, 2, ..., n)$ are called the comps of the vector $\underset{\sim}{x}$.

(A)Abstract Def. of Linear Vector Space L

(1) For every two vectors $\underset{\sim}{x}$ and $\underset{\sim}{y}$ in L, there is a sum $\underset{\sim}{x} + \underset{\sim}{y}$ in L. Vector addition is required to satisfy the following laws:

(i) $\underset{\sim}{x} + \underset{\sim}{y} = \underset{\sim}{y} + \underset{\sim}{x}$

(ii) $(\underset{\sim}{x} + \underset{\sim}{y}) + \underset{\sim}{z} = \underset{\sim}{x} + (\underset{\sim}{y} + \underset{\sim}{z})$

(iii) There exists a unique vector $\underset{\sim}{0}$ in L such that

$\underset{\sim}{x} + \underset{\sim}{0} = \underset{\sim}{x}$ for all $\underset{\sim}{x}$ in L.

(iv) For every vector $\underset{\sim}{x}$ there is a unique vector $-\underset{\sim}{x}$ such that

$\underset{\sim}{x} + (-\underset{\sim}{x}) = 0$

(2) For every α in scalar field F and every x in L there is supposed to be a vector αx in L and we require:

(i) $\alpha(\beta \underset{\sim}{x}) = (\alpha\beta)\underset{\sim}{x}$

(ii) $1\underset{\sim}{x} = \underset{\sim}{x}$

(iii) $\alpha(\underset{\sim}{x} + \underset{\sim}{y}) = \alpha\underset{\sim}{x} + \alpha\underset{\sim}{y}$

(iv) $(\alpha + \beta)\underset{\sim}{x} = \alpha\underset{\sim}{x} + \beta\underset{\sim}{x}$

(B)Linear Dependence, Independence:

Let $\underset{\sim}{u}_1, ..., \underset{\sim}{u}_n$ be vectors in L. A sum $\sum_{i=1}^{n} \alpha_i \underset{\sim}{u}_i$is call a linear combination of the vectors $\underset{\sim}{u}_i$.

These vectors are said to be linearly indep. if

$\sum_{i=1}^{n} \alpha_i \underset{\sim}{u}_i = 0$ only when all $\alpha_i = 0$—(1)

These vectors are called linearly dep. when (1) has some sol. other than α_i=0.

(C)Gramian:

Suppose that α_i do exist such that (1) is satisfied

$$\Rightarrow \alpha_1(u_1, u_1) + \alpha_2(u_1, u_2) + ... + \alpha_n(\underset{\sim}{u_1}, \underset{\sim}{u_n}) = 0$$

$$\alpha_1(\underset{\sim}{u_2}, \underset{\sim}{u_1}) + \alpha_2(\underset{\sim}{u_2}, \underset{\sim}{u_2}) + ... + \alpha_n(\underset{\sim}{u_2}, \underset{\sim}{u_n}) = 0$$

....

$$\alpha_1(\underset{\sim}{u_n}, \underset{\sim}{u_1}) + \alpha_2(\underset{\sim}{u_n}, \underset{\sim}{u_2}) + ... + \alpha_n(\underset{\sim}{u_m}, \underset{\sim}{u_n}) = 0$$

The condition for nontrivial sol. of α_i/s is

$$G(\text{Gram determinant or Gramian}) = \begin{vmatrix} (\underset{\sim}{u_1}, \underset{\sim}{u_1}) & (\underset{\sim}{u_1}, \underset{\sim}{u_2}) & \cdots & (\underset{\sim}{u_1}, \underset{\sim}{u_n}) \\ (\underset{\sim}{u_2}, \underset{\sim}{u_1}) & (\underset{\sim}{u_2}, \underset{\sim}{u_2}) & \cdots & (\underset{\sim}{u_2}, \underset{\sim}{u_n}) \\ \cdots\cdots & & & \\ (\underset{\sim}{u_n}, \underset{\sim}{u_1}) & (\underset{\sim}{u_n}, \underset{\sim}{u_2}) & \cdots & (\underset{\sim}{u_n}, \underset{\sim}{u_n}) \end{vmatrix} \equiv 0$$

Thus if a set of vectors in L are linearly dep. the Gramian must vanish. The converse can also be shown to be true:

$\Rightarrow$ A set of real vectors is linearly dep.(indep) if it's Gramian=0 ($\neq 0$)

Lemma: The vectors $\underset{\sim}{u_1}, ..., \underset{\sim}{u_m}$ are dep., if one of these vectors is some linear combination of the vectors.

(D). Basis and Dimension:

The vectors $\underset{\sim}{y_1}$,......, $\underset{\sim}{y_k}$ in L are said to span a linear subspace M if every vector $\underset{\sim}{x}$ in M is some linear combination of $\underset{\sim}{y_1}$,....., $\underset{\sim}{y_k}$.

Ex: The independent vectors

$$\underset{\sim}{y_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \underset{\sim}{y_2} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

span a certain plane where the origion in 3-space. The same plane is spanned by the depent vectors:

$$\underset{\sim}{y_1}, \underset{\sim}{y_2}, \quad \underset{\sim}{y_3} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \quad \underset{\sim}{y_4} = \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix}$$

because $\underset{\sim}{y_3} = 2\underset{\sim}{y_2} - \underset{\sim}{y_1}$ $\quad \underset{\sim}{y_4} = 3\underset{\sim}{y_3} - 2\underset{\sim}{y_1}$ are the linear combinations of $\underset{\sim}{y_1}, \underset{\sim}{y_2}$.

Basis:

Vectors $\underset{\sim}{b}_1$,......., $\underset{\sim}{b}_n$ are said to be a basis for a linear vector space L if the vectors $\underset{\sim}{b}_i$ lie in L and if they are indep., and if they span L.

Ex:

In the above example, $\underset{\sim}{y}_1$, $\underset{\sim}{y}_2$, are a basis for the plane that they span. The vectors $\underset{\sim}{y}_1$, $\underset{\sim}{y}_2$, $\underset{\sim}{y}_3$, $\underset{\sim}{y}_4$ are not a basis because they are dep.

Another basis for the plane is:

$$\underset{\sim}{z}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \underset{\sim}{z}_2 = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix}$$

$\Rightarrow$ $\underset{\sim}{z}_1$, $\underset{\sim}{z}_2$ are a basis for the plane spanned by $\underset{\sim}{y}_1$ and $\underset{\sim}{y}_2$, because $\underset{\sim}{z}_1$ and $\underset{\sim}{z}_2$ are indep. and

$$\underset{\sim}{y}_1 = \tfrac{1}{2}\underset{\sim}{z}_2 - \tfrac{3}{2}\underset{\sim}{z}_1, \qquad \underset{\sim}{y}_2 = \tfrac{1}{2}\underset{\sim}{z}_2 + \tfrac{3}{2}\underset{\sim}{z}_1$$

Thus the basis $\underset{\sim}{y}_1$, $\underset{\sim}{y}_2$ and the basis $\underset{\sim}{z}_1$, $\underset{\sim}{z}_2$ of the same plane that they span consists of two vectors

$\Rightarrow$ every basis must contain exactly the same vectors

$\Rightarrow$ we call the dimension of the space which its basis span is the number of vectors in any basis.

Theorem: Let $\underset{\sim}{a}_1$,......., $\underset{\sim}{a}_r$ be a basis a linear vector space L, let $\underset{\sim}{b}_1$,........, $\underset{\sim}{b}_s$ be another basis for L, then $r = s$.

Theorem: If m vectors lie in a linear space of dimensions $n < m$,then the m vectors are dep.

Theorem: Every set of n indep. vectors in an n-dim. linear space is a basis for the space.

(E) Gram - Schmidt Orthogonolization:

Given a set of n indep. vectors $\underset{\sim}{u}_1$,......., $\underset{\sim}{u}_n$, we can construct an orthonormal set from this indep. set.

Let $\underset{\sim}{v}_1 = \underset{\sim}{u}_1$ find $\underset{\sim}{e}_1 = \frac{\underset{\sim}{u}_1}{|\underset{\sim}{v}_1|}$

then find

$$\underset{\sim}{v}_2 = \underset{\sim}{u}_2 - (\underset{\sim}{e}_1, \underset{\sim}{u}_2)\underset{\sim}{e}_1 \quad \text{and} \quad \underset{\sim}{e}_2 = \frac{\underset{\sim}{v}_2}{|\underset{\sim}{v}_2|}$$

$$\underset{\sim}{v}_3 = \underset{\sim}{u}_3 - (\underset{\sim}{e}_1, \underset{\sim}{u}_3)\underset{\sim}{e}_1 - (\underset{\sim}{e}_2, \underset{\sim}{u}_3)\underset{\sim}{e}_2 \quad \text{and} \quad \underset{\sim}{e}_3 = \frac{\underset{\sim}{v}_3}{|\underset{\sim}{v}_3|}$$

....

$$\underset{\sim}{v}_n = \underset{\sim}{u}_n - \sum_{j=1}^{n-1}(\underset{\sim}{e}_j, \underset{\sim}{u}_n)\underset{\sim}{e}_j \quad \text{and} \quad \underset{\sim}{e}_n = \frac{\underset{\sim}{v}_n}{|\underset{\sim}{v}_n|}$$

It is easily to prove that v_1 ,.......,v_m form a mutually orthogonal set. for example:

$$(\underset{\sim}{v}_1, \underset{\sim}{v}_2) = (\underset{\sim}{u}_1, \underset{\sim}{u}_2) - (\underset{\sim}{u}_1, (\underset{\sim}{e}_1, \underset{\sim}{u}_2)\underset{\sim}{e}_1) = (\underset{\sim}{u}_1, \underset{\sim}{u}_2) - (\underset{\sim}{u}_1, \underset{\sim}{u}_2) = 0$$

thus the set of vectors $\underset{\sim}{e}_1$,......, $\underset{\sim}{e}_n$ forms an orthonormal set.

(F). Vector Expansion:

Def.: The linear vector space in n-dim. (denoted by E^n) if it processes a set of n indep. vectors, but every set of n+1 vectors is a dep. set, and we call the set of n indep. vectors forms a complete set in E^n .

A basis consisting orthonormal vectors is known as an orthonormal basis. Given a complete orthonormal basis $\underset{\sim}{e}_1$,......, $\underset{\sim}{e}_n$ in E^n space, then any vector $\underset{\sim}{x}$ in E^n can be written in the form:

$$\underset{\sim}{x} = \sum_{k=1}^{n} c_k \underset{\sim}{e}_k$$

$$\Rightarrow \quad (\underset{\sim}{x}, \underset{\sim}{e}_j) = \sum_{k=1}^{n} c_k (\underset{\sim}{e}_k, \underset{\sim}{e}_j) = \sum_{k=1}^{n} c_k \delta_{kj} = c_j$$

$$\Rightarrow \quad \underset{\sim}{x} = \sum_{k=1}^{n} (\underset{\sim}{x}, \underset{\sim}{e}_k)\underset{\sim}{e}_k$$

where $c_k = (\underset{\sim}{x}, \underset{\sim}{e}_k)$ are called the Fourier Coeffs. of $\underset{\sim}{x}$ w.r.t. the orthonormal set $\underset{\sim}{e}_k$.

(G)Bessel Inequality, Completeness Relation

When we want to approx. an arb. vector $\underset{\sim}{x}$ in E^n in terms of a linear combination of the orthonormal set $\underset{\sim}{e}_1, ..., \underset{\sim}{e}_n$ $k \leq n$.

write

$$\underset{\sim}{x} \simeq \sum_{i=1}^{k} a_i \underset{\sim}{e}_i$$

How to choose a_i to get the best approx. for $\underset{\sim}{x}$? What does the best approx. mean - we mean in the least square sense.

i.e. to choose a_i such that $\Delta = \left|\underset{\sim}{x} - \sum_{i=1}^{k} a_i \underset{\sim}{e_i}\right|^2$ is min.

now

$$\Delta = |\underset{\sim}{x}|^2 + \sum_{i=1}^{k} {a_i}^2 - 2\sum_{i=1}^{k} a_i c_i.$$

$$= |\underset{\sim}{x}|^2 - \sum_{i=1}^{k} {c_i}^2 + \sum_{i=1}^{k} (a_i - c_i)^2$$

For a given $\underset{\sim}{x}$, $\Rightarrow |\underset{\sim}{x}|^2$ and c_i are fixed. It is clear that Δ is min. when $a_i = c_i \Rightarrow$ the optional coeffs. $a_i = c_i = (\underset{\sim}{x}, \underset{\sim}{e_i})$ are the Fourier coeffs.

When $a_i = c_i \Rightarrow$

$$0 \leq \Delta = |\underset{\sim}{x}|^2 - \sum_{i=1}^{k} {c_i}^2$$

$$\Rightarrow \quad |\underset{\sim}{x}|^2 \geq \sum_{i=1}^{k} {c_i}^2 \text{ Bessel's inequality.}$$

When the value of k increases, the right hand sides increases, but is not larger than $|\underset{\sim}{x}|^2$, until k=n (i.e. the orhtonormal set is complete), then the equality holds, and thus we have the so-called completeness relation for E^n space.

$$|\underset{\sim}{x}|^2 = \sum_{i=1}^{n} {c_i}^2$$

I-3. Function space

(A) L_2[a,b] space:

Space of real fucs. $f(x)$ which is defined on [a,b] and square integrable i.e. $\|f\|^2 = (f,f) = \int_a^b f^2(x)dx < \infty$ In the language of vector space, we say that "any n linearly indep. vectors form a basis in E^n space". Similarly, in function space, it is possible to choose a set of basis function such that any fuction, satisfying appropriate condition can be expressed as a linear combination to a basis in L_2[a,b], Certainly, any such set of fucs. must have infinitely many numbers; that is, such a L_2[a,b] comprises infinitely many dimensions.

(B). Schwarz Inequality

Given $f(x)$, $g(x)$ in $L_2[a,b]$

Define $(f,g) = \int_a^b f(x)g(x)dx$ then $(f,g)^2 \leq (f,f)(g,g)$.

proof: $(f+\alpha g, f+\alpha g) = \alpha^2(g,g) + 2\alpha(f,g) + (f,f) \geq 0$

$\Rightarrow \quad (f,g)^2 - (f,f)(g,g) \leq 0$

$\Rightarrow \quad (f,g)^2 \leq (f,f)(g,g)$

(C). Linear Dependense, Independence

Criterion: A set of fucs. $\phi_1(x),..,\phi_n(x)$ in $L_2[a,b]$ is linear dep.(indep.) if its Gramian (G) vanishes (does not vanish). where

$$G = \begin{vmatrix} (\phi_1,\phi_1) & (\phi_1,\phi_2) & \cdots & (\phi_1,\phi_n) \\ (\phi_2,\phi_1) & (\phi_2,\phi_2) & \cdots & (\phi_2,\phi_n) \\ \cdots\cdots\cdots & & & \\ (\phi_n,\phi_1) & (\phi_n,\phi_2) & \cdots & (\phi_n,\phi_n) \end{vmatrix}$$

The proof is the same as in linear vector space.

(D). The orthogonal Systems

A set of real fucs. $\phi_1, ...\phi_n(x), ...$ is called an orthogonal set of fucs. in $L_2[a,b]$ if these fucs are defined in $L_2[a,b]$ if all the integrals (ϕ_m, ϕ_n) exist and are zero for all pairs of distinct fucs. in the set. i.e. $(\phi_m, \phi_n) = 0 \quad m \neq n, \quad m, n = 1, 2, ...$

If each fuc. of the set is devided by the so-called *norm*.

$$\| \phi_m \| = \sqrt{(\phi_m, \phi_m)} = \left[\int_a^b {\phi_m}^2(x) dx \right]^{1/2}$$

then the new set

$$g_m(x) = \frac{\phi_m}{\|\phi_m(x)\|} \quad \text{m=1,2,...}$$

form an orthonormal set i.e. $\delta_{mn} = (g_m, g_n)$

Some important set of real fucs. $\phi_1(x)$,..., occuring in application are not orhtogonal but have the property that for some fuc. r(x).

$$\int_a^b r(x)\phi_m(x)\phi_n(x) dx = 0 \quad \text{for} \quad m \neq n$$

Such a set is then said to be orthogonal w.r.t. the weight fuc. r(x) on [a,b].

The norm is now defined as

$$\|\phi_m\| = \left[\int_a^b r(x) {\phi_m}^2(x) dx \right]^{1/2}$$

And if the norm of each fuc. is 1, then the set is said to be orthonormal on [a,b] w.r.t. r(x).

(E). Gram-Schmidt Orthogonalization

A set of indep. fucs. $f_1(x), ..., f_m(x)$ defined on L_2[a,b] can be formed to a set of orthonormal fucs. $\phi_1(x), \phi_2(x), ...\phi_n$ i.e. $(\phi_i, \phi_j) = \delta_{ij}$

let

$g_1(x) = f_1(x)$ find $\phi_1(x) = \frac{g_1(x)}{\|g_1(x)\|}$

$g_2(x) = f_2(x) - (f_2, \phi_1)\phi_1$, find $\phi_2 = \frac{g_2(x)}{\|g_2\|}$

$g_3(x) = f_3 - (f_3, \phi_1)\phi_1 - (f_3, \phi_2)\phi_2$ find $\phi_3 = \frac{g_3(x)}{\|g_3\|}$

....

$g_m(x) = f_m(x) - \sum_{i=1}^{m-1}(f_m, \phi_i)$, find $\phi_m = \frac{g_m(x)}{\|g_m\|}$

....

(F). Bessel Inequality

Let $\phi_1(x), ..., \phi_n(x)$ be an orthonormal set of fucs. on L_2[a,b] and f be any fuc. on L_2[a,b]. The real number

$c_i = (f, \phi_i) = \int_a^b f(x)\phi_i(x)dx$ i=1,2,...,n

are called the generalized Fourier Coeffs. of $f(x)$ w.r.t. the set $\phi_1, ... \phi_n$ i=1,2,...,n

We now want to approx. $f(x)$ by the linear combination

i.e. $f(x) \cong \sum_{i=1}^n a_i\phi_i$

to choose a_i such that the mean square error is min.

$$\begin{aligned}
\Delta_n &= \int_a^b \left[f(x) - \sum_{i=1}^n a_i\phi_i(x)\right]^2 dx \\
&= \int_a^b \left[f(x)\right]^2 dx - 2\sum_{i=1}^n a_i \int_a^b f(x)\phi_i(x)dx + \int_a^b \left[\sum_{i=1}^n a_i\phi_i(x)\right]^2 dx \\
&= \|f\|^2 - 2\sum_{i=1}^n a_i c_i + \sum_{i=1}^n a_i^2 \\
&= \|f\|^2 - \sum_{i=1}^n c_i^2 + \sum_{i=1}^n (a_i - c_i)^2
\end{aligned}$$

For a given $f(x)$, then $\|f\|$ and c_i are fixed

$\Rightarrow$ Δ_n is min if $a_i = c_i$

and $\left(\Delta_n\right)_{min} = \int_a^b f(x)^2 dx - \sum_{i=1}^n c_i^2 \geq 0$

As n increases, the nonnegative $(\Delta_n)_{min}$ can only decreases. Now $\int_a^b f^2(x)dx \geq \sum_{i=1}^n c_i^2$ where n is arb. Since the integral on the LHS is bounded by a const., hence the sum on the RHS has a finite limit as $n \to \infty$

$\Rightarrow$ The series $\sum_{i=1}^\infty c_i^2$ converges and $\int_a^b f^2(x)dx \geq \sum_{k=1}^\infty c_k^2$ (B.I)

Also from the convergence of the series on the RHS

$\Rightarrow$ $\lim_{n\to\infty} c_n = 0$

i.e., Four-coeff.approaches zero as $n \to \infty$
ier

(G). Completeness Relation

Although $\sum_{i=1}^{\infty} c_i^2$ converges to some positive number, not greater than $\int_a^b f^2 dx$. However, no assurance that the limit to which this sense converges will actually equal to $\int_a^b f^2 dx$. So it is not sufficient merely have, a set of infinitely many mutually orthogonal fucs.

Ex. The fucs $\sqrt{\frac{2}{l}} \cos \frac{n\pi x}{l}$, n=1,2,... contsitute an infinite orthernormal set of fucs over [0,l] However, for the fuc. $f(x) = 1$, its fourier coeffs.

$C_n = \sqrt{\frac{2}{l}} \int_0^l 1 \times \cos \frac{n\pi x}{l} dx = 0$, n=1,2,...

but $\int_0^l f^2(x)dx = \int_0^l dx = l$

$\Rightarrow \sum_{i=1}^{\infty} C_i^2 = 0 < \int_0^l f^2 dx = l$

In a vector space of n-dim, if we construct a set of n multually orthogonal unit vectors, then the possibility of expressing any other vector as a linear combination of these vectors is a consequences of the fact that no other vector can be linear indep. of them; that is, there exists no vector in that space, other than the zero vector, which is simultaneously orthgonal to these n vectors. However, in fuc. space of infinitely many dims. the difficulty consists of the fact that a fuc. may simultaneously be orthogonal to an infinite number of multually orthogonal fucs.

Thus in this example, the fuc. $f(x) = 1$ is orthogonal to all fucs. over [0,l]. However, it can be shown that any fuc. which has this property differs from a const. by a trial fuc. so that the extended set

$\frac{1}{\sqrt{l}}, \sqrt{\frac{2}{l}} \cos \frac{\pi x}{l}, \sqrt{\frac{2}{l}} \cos \frac{2\pi x}{l}, \ldots$

there is no nontrivial fuc. whose Fourier consts all vanish. Such a set of orhtogonal fucs is said to be complete.

The system $\phi_1, \ldots \phi_n, \ldots$ is said to be complete if for any square integrable fuc. $f(x)$, the equality $\int_a^b f^2(x)dx = \sum_{i=1}^{n} c_i^2$ holds or $\lim_{n\to\infty} \int_a^b [f(x) - \sum_{i=1}^{n} c_i \phi_i(x)]^2 dx = 0$

This does not necessarily $\Rightarrow f = \sum_{i=1}^{\infty} c_i \phi_i$ is true. We know only that mean error tends to zero, and we say that if completeness relation is satisfied, then the series converges to the mean to $f(x)$ written as $f(x) = \lim_{n\to\infty} \sum_{i=1}^{n} c_i\phi_i(x)$

However, if $f(x)$ is continuous fuc. throughout the interval, and if we can prove that the series $\sum_{k=1}^{\infty} c_k\phi_k$ also represent a continuous fuc. (if ϕ_k are continuous and if the converges uniformly in the interval), then the difference between these two fucs. is a continuous fuc. with zero norm, and hence is indeed zero everywhere in (a,b), so that the series converges to $f(x)$ in the true set at each point.

I-4. Properties of Complete System

Theorem: Let $f(x), F(x)$ be defined on L_2[a,b] for which

$$f(x) = \lim_{n\to\infty} \sum_{k=1}^{n} c_k \phi_k$$
$$F(x) = \lim_{n\to\infty} \sum_{k=1}^{n} C_k \phi_k$$

Then we have $\int_a^b f(x)F(x)dx = \sum_{k=1}^{\infty} c_k C_k$

Proof:

Since f+F, and f-F are square integrable, from the completeness relation

$\Rightarrow$

$$\int_a^b [f+F]^2 dx = \sum_{k=1}^{\infty} (c_k + C_k)^2$$
$$\int_a^b [f-F]^2 dx = \sum_{k=1}^{\infty} (c_k - C_k)^2$$
$$\Rightarrow \quad 4\int_a^b f(x)F(x)dx = 4\sum_{k=1} c_k C_k$$

Theorem: Every square integrable fuc. $f(x)$ is uniquely determined (except for its value at a finite number of points) by its Frusier series.

Proof:

Suppose there are two fucs. $f(x), g(x)$ having the identical Fourier series representation

$$\text{i.e.} \lim_{n\to\infty} \int_a^b [f(x) - \sum_{k=1}^{n} c_k \phi_k(x)]^2 dx = 0$$
$$\lim_{n\to\infty} \int_a^b [g(x) - \sum_{k=1}^{n} c_k \phi_k(x)]^2 dx = 0$$

Then using $(\alpha+\beta)^2 \le 2(\alpha^2+\beta^2)$ we find

$$
\begin{aligned}
0 &\le \int_a^b [g(x) - f(x)]^2 dx \\
&= \int_a^b [\Big(g(x) - \sum c_k\phi_k\Big) + \Big(\sum c_k\phi_k - f(x)\Big)]^2 dx \\
&\le 2\int_a^b [g - \sum_{k=1}^n c_k\phi_k]^2 dx + 2\int_a^b [f(x) - \sum_{k=1}^n c_k\phi_k]^2 dx \\
&= 0 \quad (\text{by} \quad \lim_{n\to\infty})
\end{aligned}
$$

$$
\Rightarrow \int_a^b [g(x) - f(x)]^2 = 0
$$

$\Rightarrow g(x) = f(x)$ at the pts of continuity of the integrand

$\Rightarrow g(x)$, and $f(x)$ coincide everywhere, except possibly at a finite number of pts. of discontinuity.

Theorem: An continuous fuc. $f(x)$ which is orthogonal to all the fucs. of the complete system must be identically zero.

Proof:

Since $C_i = (f, \phi_i) = 0$, i=1,2,.......

$\Rightarrow \quad \int_a^b f^2(x)dx = 0$

$\Rightarrow f(x) \equiv 0 \quad$ since f is continuous

Theorem: For any continuous fuc. $f(x)$, its Fourier series converges to $f(x)$ in the true sense at each point.

Proof: Since

$$
\lim_{n\to\infty} \int_a^b [f(x) - \sum_{k=1}^n c_k\phi_k(x)]^2 dx = 0
$$

$$
\text{and let} \quad g(x) = \sum_{k=1}^\infty c_k\phi_k(x)
$$

we can prove $f(x) = g(x)$ at every point.

$$
\Rightarrow \quad f(x) = \sum_{k=1}^\infty c_k\phi_k(x).
$$

Theorem: The Fourier series of every square integrable fuc. $f(x)$ can be integrated term by term.

In other words, if

$$f(x) \sim c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \ldots..$$

then $\int_{x_1}^{x_2} f(x)dx = c_1 \int_{x_1}^{x_2} \phi_1(x)dx + \ldots + c_n \int_{x_1}^{x_2} \phi_n(x)dx + \ldots$ where x_1, x_2 are any points on the inteval [a,b].

Proof:

Assume $x_2 > x_1$

$$\left|\int_{x_1}^{x_2} f dx - \sum_{k=1}^{n} c_k \int_{x_1}^{x_2} \phi_k dx\right|$$
$$\leq \int_{x_1}^{x_2} |f - \sum_{k=1}^{n} c_k\phi_k| dx$$
$$\leq \int_a^b |f - \sum_{k=1}^{n} c_k\phi_k| dx$$
$$\leq \sqrt{\int_a^b [f - \sum_{k=1}^{n} c_k\phi_k]^2 dx \int_a^b 1 dx}$$

(by Schwarz ineq.)

Take $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} \left|\int_{x_1}^{x_2} f dx - \sum_{k=1}^{n} c_k \int_{x_1}^{x_2} \phi_k(x)dx\right| = 0$$

I-5. Sturm-Liouville System

(A)Self-Adjoint Operator

Consider a general 2nd o.d.e.

$$Lu(x) = a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u = 0$$

i.e. $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c$

From

$$\int_{a_1}^{a_2} vLudx = \int_{a_1}^{a_2} (vau'' + vbu' + vcu)dx$$
$$= [avu' - (av)'u + bvu]_{a_1}^{a_2} + \int_{a_1}^{a_2} u[(av)'' - (bv)' + cv]dx$$
$$= [..........]_{a_1}^{a_2} + \int_{a_1}^{a_2} uL^*vdx$$

where $L^*v = (av)'' - (bv)' + cv$ then L^* is called the adjoint operator associated with I and is

$$L^* = a\frac{d^2}{dx^2} + (2a' - b)\frac{d}{dx} + (a'' - b' + c)$$

In the event that $L^* = L$, we say L is self-adjoint. It is clear to see if $b = 2a' - b$, $c = a'' - b' + c$ then L is self-adjoint.

These conditions are satisfied if $b = a'$. So that, a linear, 2nd order, self-adj. o.d.e can be expressed in the form

$$L = p(x)\frac{d^2}{dx^2} + p'(x)\frac{d}{dx} + q(x)$$
$$= \frac{d}{dx}[p(x)\frac{d}{dx}] + q(x)$$

Actually, any linear, 2nd order o.d.e.

$$Lu = a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u(x) = 0 \quad —(1)$$

can be converted to an self-adj. o.d.e by multiplying the integrating factor

$I(x) = exp[\int^x \frac{b(\eta)}{a(\eta)} d\eta]$

$(1) \Rightarrow$

$I(x)\frac{d^2u}{dx^2} + \frac{b(x)}{a(x)}I(x)\frac{du}{dx} + \frac{c(x)}{a(x)}I(x)u(x) = 0$

$\Rightarrow \frac{d}{dx}[I(x)\frac{du}{dx}] + \frac{c(x)}{a(x)}I(x)u(x) = 0$

or $\frac{d}{dx}[p(x)\frac{du}{dx}] + q(x)u(x) = 0$

where $p(x) = I(x) = exp[\int^x \frac{b(\eta)}{a(\eta)} d\eta]$, $q(x) = \frac{c(x)}{a(x)}I(x)$

(B)Regular S-L problem

(1)Regular S-L equation

$$L[u] + \lambda r(x)u(x) = 0 \quad \text{———}(2)$$

if $L = \frac{d}{dx}[p(x)\frac{d}{dx}] - q(x)$, λ is a real parameter $q(x) \geq 0, r(x) > 0$ are continuous of $[a, b]$ $p(x) > 0$ is continuously differentiable. Then (2) is called a regular S-L equation.

Ex.1: Bessel eq. of order n

$$u'' + \frac{1}{x}u' + (\alpha^2 - \frac{n^2}{x^2})u = 0 \quad \text{on} \quad 1 < x < a$$
$$\Rightarrow \frac{d}{dx}(x\frac{du}{dx}) + (-\frac{n^2}{x} + \alpha^2 x)u = 0$$
$$\Rightarrow p(x) = x, q(x) = \frac{n^2}{x}, \lambda = \alpha^2, r(x) = x$$

Ex.2: Legendre's eq. on $-1 < x < 1$

$(1 - x^2)u'' - 2xu' + \alpha(\alpha + 1)u = 0$

$\Rightarrow \frac{d}{dx}[(1 - x^2)\frac{du}{dx}] + \alpha(\alpha + 1)u = 0$

$\Rightarrow p(x) = 1 - x^2, q(x) = 0, \lambda = \alpha(\alpha + 1), r(x) = 1$

Ex.3

$u'' + u' + \alpha^2 u = 0 \quad \text{on} \quad 0 < x < l$

$I = exp[\int^x d\eta] = e^x$

$\Rightarrow \frac{d}{dx}(e^x \frac{du}{dx}) + \alpha^2 e^x u = 0$

$\Rightarrow p(x) = e^x, q(x) = 0, \lambda = \alpha^2, r(x) = e^x.$

(2)General homogeneous b.c's

$$B_a[u] = \alpha_1 u(a) + \beta_1 u'(a) = 0$$

$$B_b[u] = \alpha_2 u(b) + \beta_2 u'(b) = 0$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2$ are constants.

* S-L problem: prob. of determining the dependence of u on and the parameter λ and the dependence of λ on the homo. $b.c$'s imposed on u.
* The value of λ which yield the nontrivial sol. such that (2) is satisfied are called the eigenvalues, say $\lambda_1 \lambda_2 ...$ Corresponding to each eigenvalue the nontrivial sol. is called the eigenfuc.

(C)S-L Theorems

For a regular S-L problems.

(1). Eigenfucs corresponding to distinct eigenvalues are orthogonal w.r.t. the weight fuc. $r(x)$.

Proof: Say

$$Lu_i + \lambda_i r u_i = 0$$

$$Lu_j + \lambda_j r u_j = 0$$

$$\Rightarrow (u_j, Lu_i) - (u_i, Lu_j) = (\lambda_j - \lambda_i)(ru_i, u_j)$$

now

$$LHS = \int_a^b \left\{ u_j[(pu_i')' - q(u_i)] - u_i[(pu_j')' - qu_j] \right\} dx$$

$$= (pu_ju_i' - pu_iu_j')|_a^b - \int_a^b (u_j'pu_i' - u_i'pu_j')dx$$

$$= p(b)[u_j(b)u_i'(b) - u_i(b)u_j'(b)] - p(a)[u_j(a)u_i'(a) - u_i(a)u_j'(a)]$$

$$= \triangle$$

now from the b.c's

$$\begin{cases} \alpha_1 u_i(a) + \beta_1 u_i'(a) = 0 \\ \alpha_1 u_j(a) + \beta_1 u_j'(a) = 0 \end{cases}$$

$$\Rightarrow u_i(a)u_j'(a) - u_i'(a)u_j(a) = 0$$

$$\begin{cases} \alpha_2 u_i(b) + \beta_2 u_i'(b) = 0 \\ \alpha_2 u_j(b) + \beta_2 u_j'(b) = 0 \end{cases}$$

$$\Rightarrow u_i(b)u_j'(b) - u_i'(b)u_j(b) = 0$$

$$\Rightarrow \triangle = 0$$

i.e. $(u_j, Lu_i) = (u_i, Lu_j) \Rightarrow (\lambda_j - \lambda_i)(ru_i, u_j) = 0$ —(3)

but $\lambda_i \neq \lambda_j \Rightarrow \int_a^b r(x)u_i(x)u_j(x)dx = 0$

(2). If $r(x) \neq 0$ in $a < x < b$, the eigenvalues of the S-L problem are all real.

Proof:

suppose $\lambda_n = \alpha + i\beta$ where α, β are real constants with corresponding u_n. i.e. $(pu_n')' + (-q + \lambda_n r)u_n = 0$

If $u_n = \varphi + i\psi$ then

$$\Rightarrow (p\varphi')' + (-q + \alpha r)\varphi - \beta r\psi + i[(p\psi')' + (-q + \alpha r)\psi + \beta r\varphi] = 0$$

or $(p\varphi')' + (-q + \alpha r)\varphi - \beta r\psi = 0$

$$(p\psi')' + (-q + \alpha r)\psi - \beta r\varphi = 0$$

The complex conjugates $\lambda_m = \overline{\lambda_n} = \alpha - i\beta$ and $u_m = \overline{u_n} = \varphi - i\psi$ are also a distinct eigenvalue and corresponding eigenfuc.

(3) $\Rightarrow (\lambda_m - \lambda_n)\int_a^b r(x)u_n(x)u_m(x) = 0$ dx

or $2i\beta \int_a^b r(x)(\varphi^2 + \psi^2)dx = 0$

since $r(x)(\varphi^2 + \psi^2) \neq 0$ in (a, b)

$\Rightarrow \beta = 0 \quad \Rightarrow \quad \lambda_n$ is real.

(3). There is an infinite discrete set of $\lambda_1, < \lambda_2, < ..., < \lambda_n \quad n \to \infty$ with corresponding eigenfucs $u_1, u_2, ..., u_n, ...$

proof: Define

$$Q(f,g) = \int_a^b (pf'g' + qfg)dx + [\alpha_i pfg]_a^b \quad where \quad \alpha_i \quad \text{denotes the b.c. } u' + \alpha_i u = 0$$

$$N(f,g) = \int_a^b rfgdx$$

and write

$$Q(f,f) = Q(f) = \int_a^b (pf'^2 + qf^2)dx + [\alpha_i pf^2]_a^b$$

N(f,f)=N(f)= $\int_a^b rf^2 dx$

We will prove first that the min. of $\frac{Q}{N}$ will be equal to the lowest eigenvalue λ_1, and the corresponding minimizing fuc. will be the first eigenfuc u_1.

Assume that there exists a sol. u_1 to the first min. prob. i.e. $Q(u_1) = \lambda_1 N(u_1)$ then $Q(u_1 + \epsilon\eta) \geq \lambda_1 N(u_1 + \epsilon\eta)$

where ϵ is an arb. real number and η is an arb. fuc

$$\Rightarrow \mathrm{Q(u_1)} + 2\epsilon Q(u_1,\eta) + \epsilon^2 Q(\eta) \geq \lambda_1[N(u_1) + 2\epsilon N(u_1,\eta) + \epsilon^2 N(\eta)]$$

$$\Rightarrow 2\epsilon[Q(u_1,\eta) - \lambda_1 N(u_1,\eta)] + \epsilon^2[Q(\eta) - \lambda_1 N(\eta)] \geq 0$$

The eq. must be true for arb. ϵ

$$\Rightarrow Q(u_1,\eta) - \lambda_1 N(u_1,\eta) = 0$$

$$\Rightarrow \int_a^b (pu_1'\eta' + qu_1\eta)dx + [\alpha_i pu_1\eta]_a^b - \lambda_1 \int_a^b ru_1\eta dx = 0$$

$$[p(\alpha u_1 + u_1')\eta]_a^b - \int_a^b [(pu_1')' - qu_1 + \lambda_1 ru_1]\eta dx = 0$$

since η(x) is arb.

$\Rightarrow$ The 1st min prob of $\frac{Q}{N}$ corresponds to the b.v.p

$Lu_1 + \lambda_1 r u_1 = 0$ in (a, b)

$\alpha_i u_1 + u_1{}' = 0$ on the pts. a and b

next, we min $Q(f)/N(f)$ subject to $\int_a^b r f u_1(x) dx = 0$. This minimum will be equal to the next eigenvalue λ_2, and the minimizing fuc. will be the 2nd eigenfuc. u_2.

Assume the prob. has a sol. u_2 i.e. $Q(u_2) = \lambda_2 N(u_2)$, $N(u_1, u_2) = 0$ but this time, since $N(u_1, u_2) = 0 \Rightarrow N(u_1, \eta) = 0$ hence $\eta(x)$ is not so arbitrary.

Let's state with an arb. admissible fuc. $\zeta(x)$ let $\eta = \zeta - cu_1$, i.e. $\zeta = \eta + cu_1$, where c$=\frac{N(u_1,\zeta)}{N(u_1)}$

In this way,

$$N(u_1, \eta) = N(u_1, \zeta - cu_1)$$
$$= N(u_1, \zeta) - cN(u_1, u_1)$$
$$= cN(u_1) - cN(u_1) = 0$$

Thus we have

$$Q(u_2 + \epsilon\zeta - \epsilon c u_1) \geq \lambda_2 N(u_2 + \epsilon\zeta - \epsilon c u_1)$$

By expansion $\Rightarrow$

$$Q(u_2 + \epsilon^2 Q(\zeta) + \epsilon^2 c^2 Q(u_1) + 2\epsilon Q(u_2, \zeta) - 2\epsilon c Q(u_2, u_1) - 2\epsilon^2 c Q(\zeta, u_1)$$
$$\geq \lambda_2 [N(u_2) + \epsilon^2 N(\zeta) + \epsilon^2 c^2 N(u_1) + 2\epsilon N(u_2, \zeta) - 2\epsilon c N(u_2, u_1) - 2\epsilon^2 c N(\zeta, u_1)]$$

since $Q(u_1) = \lambda_1 N(u_1)$

$$Q(\zeta, u_1) = \lambda_1 N(\zeta, u_1) = \lambda_1 c N(u_1)$$

and $N(\zeta, u_1) = cN(u_1)$

hence

$$2\epsilon[Q(u_2, \zeta) - \lambda_2 N(u_2, \zeta) - c\{Q(u_2, u_1) - \lambda_2 N(u_2, u_1)\}]$$
$$+\epsilon^2 [Q(\zeta) - \lambda_2 N(\zeta) + c^2(\lambda_1 - \lambda_2) N(u_1)] \geq 0$$
$$\Rightarrow Q(u_2, \zeta) - \lambda_2 N(u_2, \zeta) - c\{Q(u_2, u_1) - \lambda_2 N(u_2, u_1)\} = 0$$

now since $N(u_2, u_1) = 0$ and $Q(u_1, \zeta) - \lambda_1 N(u_1, \zeta) = 0$ so that $\zeta = u_2 \Rightarrow Q(u_2, u_1) = 0$

Hence $Q(u_2, \zeta) - \lambda_2 N(u_2, \zeta) = 0$ with ζ arb.

$\Rightarrow$ The 2nd mim. prob. corresponds to

$$\begin{aligned} Eq: &\quad Lu_2 + \lambda_2 r u_2 = 0 \\ +b.c.'s: &\quad \alpha u_2 + u_2' = 0 \end{aligned}$$

We can continue in this way to generate successive eigenvalues and eigenfucs.

(4). If $f(x)$ is any continuous fuc. and has piecewise continuous 1st and 2nd derivatives, and satisfies the homo. $b.c's$ at a and b, then it has a uniformly and absolutely convergent generalized Fourier series expansion in the S-L eigenfucs.: Eigenfucs. form a complete orthogonal set in the space of piecewise smooth fucs.

Proof:

If the expansion $f(x) = \lim_{n\to\infty} \sum_{i=1}^{\infty} c_i u_i(x)$ exists and is convergent, then the difference

$f_n(x) = f(x) - \sum_{i=1}^{n-1} c_i u_i(x)$

must apperoach zero (on the average) as $n \to \infty$

i.e. $\triangle_n{}^2 = \int_a^b r(x) f_n{}^2(x) dx \quad \xrightarrow{\text{as} n\to\infty} \quad 0$

where

$$c_i = \frac{1}{E_i{}^2} \int_a^b r f(x) u_i(x) dx E_i^2 = \int_a^b r u_i{}^2(x) dx$$

Define the fuc $F_n(x) = \frac{f_n(x)}{\triangle_n}$ which has the property $\int_a^b r(x) F_n^2(x) dx = 1$ and

$\int_a^b r(x) u_m(x) F_n(x) dx = \int_a^b r(x) u_m(x) \frac{1}{\triangle_n}(f - \sum_{i=1}^{n-1} c_i u_i) dx$

$= \begin{cases} \frac{E_m{}^2}{\triangle_n}(c_m - c_m) = 0 & \text{when } m \le n-1 \\ \frac{E_m{}^2}{\triangle_n} c_m & \text{when } m > n-1 \end{cases}$

which possesses the properties of eigenfucs

Thus

$$\begin{aligned} Q(F_n) &= \int_a^b [p(\tfrac{f_n'}{\triangle_n})^2 + q(\tfrac{f_n}{\triangle_n})^2] dx + [\alpha_i p(\tfrac{f_n}{\triangle_n})^2]_a^b \\ &= \tfrac{1}{\triangle_n{}^2} \int_a^b [p(f' - \textstyle\sum_{i=1}^{n-1} c_i u_i')^2 + q(f - \sum_{i=1}^{n-1} c_i u_i)^2] dx \\ &\quad + [\tfrac{\alpha_i}{\triangle_n{}^2} p(f - \textstyle\sum_{i=1}^{n-1} c_i u_i)^2]_a^b \\ &= \tfrac{1}{\triangle_n{}^2} \int_a^b (pf'^2 + qf^2) dx + [\tfrac{\alpha_i}{\triangle_n{}^2} p f^2]_a^b \end{aligned}$$

$-\frac{2}{\Delta_n{}^2}\{\int_a^b \sum_{i=1}^{n-1} c_i(pf'u_i' + qfu_i)dx + [\alpha_i p \sum_{i=1}^{n-1} c_i f u_i]_a^b\}$

$+\frac{1}{\Delta_n{}^2}\{\int_a^b \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_i c_j (pu_i'u_j' + qu_iu_j)dx + [\alpha p \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_i c_j u_i u_j]_a^b\}$

The 1st row: $= \frac{Q(f)}{\Delta_n{}^2}$

The 2nd row:

$= -\frac{2}{\Delta_n{}^2}[p \sum_{i=1}^{n-1} c_i f(\alpha_i u_i + u_i')]_a^b$

$+\frac{2}{\Delta_n{}^2}\int_a^b f \sum_{i=1}^{n-1} c_i[(pu_i')' - qu_i]dx$

$= -\frac{2}{\Delta_n{}^2}\int_a^b \sum_{i=1}^{n-1} c_i \lambda_i r f u_i dx = -\frac{2}{\Delta_n{}^2}\sum_{i=1}^{n-1} c_i{}^2 \lambda_i E_i{}^2$

The 3rd row: $= \frac{1}{\Delta_n{}^2}\sum_{i=1}^{n-1} c_i{}^2 \lambda_i E_i{}^2$

So that

$$Q(F_n) = \frac{1}{\Delta_n{}^2}[Q(f) - \sum_{i=1}^{n-1} E_i{}^2 c_i{}^2 \lambda_i]$$

The above equality is valid for any n when $n \to \infty, \lambda \to \infty$ hence $\Delta_n^2 \overset{\text{as} n \to \infty}{\longrightarrow} 0$

$\Rightarrow$ prove that the expansion is a least square fit for the fuc. f.

$\Rightarrow$ prove the completeness and the uniform and absolute convergence.

Ex: vibration of string with density $\propto \frac{1}{(1+x)^2}$

$\frac{1}{(1+x)^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad$ for $0 < x < 1 \quad , \quad t > 0$

$u(x,0) = f(x) \quad 0 \le x \le 1$

$\frac{\partial u}{\partial t}(x,0) = 0 \quad 0 \le x \le 1$

u(0,t)=0

u(1,t)=0

Sol.: Let $u(x,t) = X(x)T(t)$

$\Rightarrow X'' + \frac{\lambda}{(1+x)^2}X = 0$

$\ddot{T} + \lambda T = 0$

sol. of X has the form $(1+x)^a$

$\Rightarrow a(a-1) + \lambda = 0$

i.e. $a = \frac{1}{2}(1 \pm \sqrt{1-4\lambda})$

$X(x) = c_1(1+x)^{\frac{1}{2}(1+\sqrt{1-4\lambda})} + c_2(1+x)^{\frac{1}{2}(1-\sqrt{1-4\lambda})}$

For X(0)=0 $\Rightarrow c_1 + c_2 = 0$

$$X(1)=0 \Rightarrow c_1 2^{\frac{1}{2}(1+\sqrt{1-4\lambda})} + c_2 2^{\frac{1}{2}(1-\sqrt{1-4\lambda})} = 0$$

For the nontrivial sol of $c_1, c_2 \Rightarrow$

$$2^{\frac{1}{2}(1+\sqrt{1-4\lambda})} - 2^{\frac{1}{2}(1-\sqrt{1-4\lambda})} = 0$$

$$\Rightarrow 2^{\frac{1}{2}(1+\sqrt{1-4\lambda})}(1 - 2^{\sqrt{1-4\lambda}}) = 0$$

$$\Rightarrow 2^{\sqrt{1-4\lambda}} = 1$$

(i). $\lambda < \frac{1}{4}$,

$\Rightarrow \sqrt{1-4\lambda}$ is real.

$\Rightarrow$ no sol of λ

(ii). $\lambda = \frac{1}{4}$, two indep sols are

$(1+x)^{\frac{1}{2}}, (1+x)^{\frac{1}{2}} \log(1+x)$?

$X(0) = 0 \Rightarrow \quad c_1 = 0$

$X(1) = 0 \Rightarrow c_2 2^{\frac{1}{2}} \log 2 = 0 \Rightarrow c_2 = 0$ only trivial sol.

$\Rightarrow \lambda = \frac{1}{4}$is not an eigenvalue.

(iii). $\lambda > \frac{1}{4}, \sqrt{1-4\lambda} = 2i\sqrt{\lambda - \frac{1}{4}}$

$$(1+x)^{\frac{1}{2}(1+2i\sqrt{\lambda-\frac{1}{4}})} = (1+x)^{\frac{1}{2}} exp(i\sqrt{\lambda - \frac{1}{4}}\log(1+x))$$

$$= (1+x)^{\frac{1}{2}}[\cos(\sqrt{\lambda - \frac{1}{4}}\log(1+x)) + i\sin(\sqrt{\lambda - \frac{1}{4}}\log(1+x))]$$

Hence

$$X(x) = (1+x)^{\frac{1}{2}}[c_1 \cos(\sqrt{\lambda - \frac{1}{4}}\log(1+x)) + c_2 \sin(\sqrt{\lambda - \frac{1}{4}}\log(1+x))]$$

For $X(0) = 0 \Rightarrow c_1 = 0$

$$X(1) = 0 \Rightarrow 2^{\frac{1}{2}} c_2 \sin(\sqrt{\lambda - \frac{1}{4}}\log 2) = 0$$

$$c_2 \neq 0 \Rightarrow \sin(\sqrt{\lambda - \frac{1}{4}}\log 2) = 0$$

$$i.e. \sqrt{\lambda_n - \frac{1}{4}}\log 2 = n\pi \quad n = 1, 2, \ldots$$

$$\Rightarrow \lambda_n = \frac{n^2\pi^2}{(\log 2)^2} + \frac{1}{4}$$

therefore $X_n(x) = (1+x)^{\frac{1}{2}} \sin(\frac{\log(1+x)}{\log 2} n\pi)$

If a fuc. is expanded as

$$f(x) \sim \sum_{n=1}^{\infty} c_n X_n(x)$$

$$c_n = \frac{\int_0^1 f(x)(1+x)^{-3/2}\sin(\frac{\log(1+x)}{\log 2}n\pi)dx}{\int_0^1 (1+x)^{-1}\sin^2(\frac{\log(1+x)}{\log 2}n\pi)dx}$$

$$= \frac{2}{\log 2}\int_0^1 f(x)(1+x)^{\frac{-3}{2}}\sin(\frac{\log(1+x)}{\log 2}n\pi)dx$$

I-6. Some Extensions of S-L Theorem.

(A)For periodic boundary conditions.

$$\frac{d}{dx}[p(x)\frac{du}{dx}] + [-q(x) + \lambda r(x)]u = 0 \quad a < x < b$$

$$u(a) = u(b)$$

$$u'(a) = u'(b) \qquad (1)$$

$$r > 0 \quad , \quad p > 0 \quad \text{on} \quad a \leq b$$

Theorem:

There exists for the system a spectrum of eigenvalue λ such that

$$\lambda_1 < \lambda_2 < ... < \lambda_n \to \infty \quad \text{as} \quad n \to \infty$$

and there exists an infinite set of corresponding eigenfucs u_{11}, u_{12},u_{21}, u_{22},...,u_{n1}, u_{n2},... on $a \leq x \leq b$ such that there at least one and perhaps two linearly indep. eigenfucs u_{ni}, $i = 1$ or 2 for λ_n.

Proof:

Assume that λ_n is an eigenvalue of (1) and u_{n1} and u_{n2} are two eigenfucs for λ_n.

The wronskian of

$$W_n(x) = u_{n2}'(x)u_{n1}(x) - u_{n1}'(x)u_{n2}(x)$$

For the boundary at $s = a, b$ in general

$$W_n(a) = u_{n2}'(a)u_{n1}(a) - u_{n1}'(a)u_{n2}(a) = f_n$$

$$W_n(b) = u_{n2}'(b)u_{n1}(b) - u_{n1}'(b)u_{n2}(b) = g_n$$

For the homo. b.c.'s $f_n = g_n = 0$

$\Rightarrow$ u_{n1} and u_{n2} are dep, we only have one indep. eigenfuction corresponding to each eigenvalue.

However, for the periodic b.c's, it yields no information on the values of f_n or g_n. If $f_n \neq 0$ and $g_n \neq 0, u_{n1}$ and $u_n 2$ exist and are linearly indep.

Ex.:

$$u'' + \lambda^2 u = 0 \quad -1 < x < 1$$

$$u(-1) = u(1)$$

$$u'(-1) = u'(1)$$

general solution

$$u(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$$

for the b.c's $\Rightarrow$

$$2c_2 \sin \lambda = 0$$

$$2c_1 \lambda \sin \lambda = 0$$

For $\sin \lambda = 0$, c_1, c_2 can be both arbitrary.

$$\lambda_n = n\pi \qquad n = 0, 1, 2, 3, ...$$

for $n = 0$, ${\lambda_0}^2 = 0$, $\Rightarrow$ only one eigenfuc. , $n \geq 1, {\lambda_n}^2 = (n\pi)^2$ there correspond two i^{th} eigenfucs.

$$u_{n1} = \cos n\pi x$$

$$u_{n2} = \sin n\pi x$$

Theorem:

The eigenfucs. u_{mi} and u_{nj} corresponding to the distinct eigenvalue λ_m and λ_n are orthogonal w.r.t. $r(x)$. In addition, a linear combination u_{n2} of the two indep. fucs. u_{n1} and u_{n2} corresponding to the same λ_n can be constructed so that the indep. fucs. u_{m1} and u_{n2} are orthogonal w.r.t. r also.

Proof

From $(\lambda_m - \lambda_n) \int_a^b r(x) u_{mi}(x) u_{nj}(x) dx = 0$

for $i, j = 1, 2$, and $m \neq n, \Rightarrow \lambda_m \neq \lambda_n$

$\Rightarrow \int_a^b r u_{mi} u_{nj} dx = 0$

(B).Oscillation Theorem:

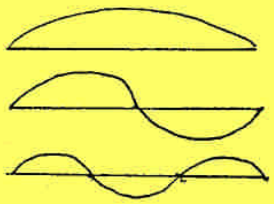
The first eigenfuc. does not vanish in (a,b). The nth eigenfuc has exactly (n-1) zero in (a,b)

Ex:

$$u'' + \lambda u = 0 \quad for \quad 0 < x < 1$$

$$u(0) = u(1) = 0$$

$$\Rightarrow u_n(x) = \sin n\pi x$$



(C)Monotonically Theorem

Reducing the interval (a, b), increasing p or q, or decreasing $r(x)$ increasing all the eigenvalues.

Let λ_1 be the lowest eigenvalues of

$$(pu')' - qu + \lambda ru = 0 \quad for \quad a < x < b$$

$$u'(a) + \alpha_1 u(a) = 0$$

$$u'(b) + \alpha_2 u(b) = 0$$

Then $\lambda_1 = min \frac{Q}{N} = min \frac{\int_a^b (p\phi'^2 + q\phi^2)dx + [\alpha_i p\phi^2]_a^b}{\int_a^b r\phi^2 dx} \quad --- (2)$

Let $\bar{\lambda}$ be the lowest eigenvalue of

$$(\bar{p}\bar{u}')' - \bar{q}\bar{u} + \bar{\lambda}\bar{r}\bar{u} = 0 \quad for \quad \bar{a} < x < \bar{b}$$

$$\bar{u}'(\bar{a}) + \bar{\alpha}_1 \bar{u}(\bar{a}) = 0$$

$$\bar{u}'(\bar{b}) + \bar{\alpha}_2 \bar{u}(\bar{b}) = 0 \quad where \quad a < \bar{a} < \bar{b} < b$$

suppose that

$$\bar{p} \geq p, \bar{q} \geq q, \bar{r} \geq r, \bar{\alpha}_1 \geq \alpha$$

Let us choose the trial fuc.

$$\phi(x) = \begin{cases} 0 & \text{for a} \leq x \leq \bar{a} \\ \bar{u}_1(x) & \text{for } \bar{a} \leq x \leq \bar{b} \\ 0 & \text{for } \bar{b} \leq x \leq b \end{cases}$$

$(2) \Rightarrow$

$$\lambda_1 = \frac{\int_a^b (p\phi\prime^2 + q\phi^2)dx + [\alpha_i p\phi^2]_a^b}{\int_a^b r\phi^2 dx}$$
$$\leq \frac{\int_{\bar{a}}^{\bar{b}} (\bar{p}\bar{u}_1' + \bar{q}\bar{u}_1^2)dx + [\bar{\alpha}_i \bar{p}\bar{u}_1^2]_a^b}{\int_a^b \bar{r}\bar{u}_1^2 dx} = \bar{\lambda}_1 \quad Q.E.D.$$

Ex:

$$u'' - \alpha u + \lambda u = 0$$
$$u(a) = u(b) = 0$$
$$\Rightarrow \lambda_n = \alpha + \frac{n^2\pi^2}{(b-a)^2}$$

it increases with increasing $\overline{\alpha(q)}$ or decreasing (b-a)

I-7 Foruier Series

Consider a b.v.p.

$$y'' + \lambda^2 y = 0$$

with periodic b.c.'s:

$$y(a) = y(a+T)$$

$$y'(a) = y'(a+T)$$

g.s.:

$$y(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$$

Apply $b.c$'s $\Rightarrow$

$$c_1 \cos \lambda a + c_2 \sin \lambda a = c_1 \cos \lambda(a+T) + c_2 \sin \lambda(a+T)$$

$$-c_1 \sin \lambda a + c_2 \cos \lambda a = -c_1 \sin \lambda(a+T) + c_2 \cos \lambda(a+T)$$

For nontrivial slooution of $c_1, c_2 \Rightarrow$

$[\cos\lambda a - \cos \lambda(a+T)]^2 + [\sin \lambda a - \sin \lambda(a+T)]^2 = 0$

$\Rightarrow \cos \lambda a = \cos \lambda(a+T)$

$\Rightarrow \lambda_n T = 2n\pi$

$\Rightarrow {\lambda_n}^2 = \frac{4n^2\pi^2}{T^2} \quad n = 0, 1, 2, 3, \ldots$

also from $\sin \lambda a = \sin \lambda(a+T)$ we have also ${\lambda_n}^2 = \frac{4n^2\pi^2}{T^2}$

Corresponding the eigenvalue λ_n, the eigenfuc is

$\phi_n = c_n \sin \lambda_n x + d_n \cos \lambda_n x$

since

$\int_a^{a+T} \sin \lambda_m x \sin \lambda_n x dx = 0 \quad for \quad m \neq n$

$\int_a^{a+T} \cos \lambda_m x \cos \lambda_n x dx = 0$ for m $\neq n$

and

$\int_a^{a+T} \sin \lambda_m x \cos \lambda_n x dx = 0$ for all m,n

$\Rightarrow \int_a^{a+T} \phi_m(x)\phi_n(x)dx = 0$ for m $\neq$ n

$\Rightarrow \phi_n(x)$ forms a complete orthogonal set

⇒ If a fuc. f(x) can be represented as

$$f(x) = A_0 + \sum_{n=1}^{\infty}(A_n \cos\lambda_n x + B_n \sin\lambda_n x) - - - (2)$$

then

$$A_0 = \frac{1}{T}\int_a^{a+T} f(x)dx$$

$$A_n = \frac{2}{T}\int_a^{a+T} f(x)\cos\lambda_n x dx$$

$$B_n = \frac{2}{T}\int_a^{a+T} f(x)\sin\lambda_n x dx$$

The series (2) is known as the complete Fourier series representation of f(x) in the interva $(a, a+T)$. Also if $f(x)$ is periodic, of period T, the series(2) represents $f(x)$ everywhere.

If $T = 2l$, then

$$(2) \Rightarrow f(x) = A_0 + \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}) - - - (3)$$

with

$$A_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx$$

$$\begin{Bmatrix} A_n \\ B_n \end{Bmatrix} = \frac{1}{l}\int_{-l}^{l} f(x)\begin{Bmatrix} \cos\frac{n\pi x}{l} \\ \sin\frac{n\pi x}{l} \end{Bmatrix} dx$$

The series (3) represents $f(x)$ in the interval $(-l, l)$, also, if $f(x)$ is periodic, or period 2 it represents $f(x)$ everywhere.

If T=2π, then

$$(2) \Rightarrow f(x) = A_0 + \sum_{n=1}^{+\infty}(A_n \cos nx + B_n \sin nx) - - - (4)$$

with

$$A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$\begin{Bmatrix} A_n \\ B_n \end{Bmatrix} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\begin{Bmatrix} \cos nx \\ \sin nx \end{Bmatrix} dx$$

series (4) represents $f(x)$ in $(-\pi, \pi)$ and represents $f(x)$ everywhere if $f(x)$ is periodic.

Ex.: Heat Conduction in Slab.

$$u_t = \alpha u_{xx} \qquad 0 < x < l \quad , \quad t > 0$$

I.C.:

$$u(x,0) = u_0$$

$$u(0,t) = u_1$$

$$u(l,t) = u_2$$

By the method of sep. of vars.

$$u(x,t) = X(x)T(t)$$

$$\Rightarrow \frac{\dot{T}}{\alpha T} = \frac{X''}{X} = -\lambda^2 \quad (\lambda : real)$$

$$\Rightarrow \dot{T} + \lambda^2 \alpha T = 0 \leftrightarrow T \sim e^{-\alpha\lambda^2 t}$$

$$X'' + \lambda^2 X = 0 \leftrightarrow X \sim A\cos\lambda x + B\sin\lambda x$$

therefore the general sol:

$$u(x,t) = \sum_{n=0}^{\infty} e^{-\alpha\lambda_n{}^2 x}(A_n \cos\alpha_n x + B_n \sin\lambda_n x)$$

However, with the inhomo., b.c's, it is difficult to find the eigenvalues. By the principle of superposition, we separate the solution in steady and transient state i.e. let $u(x,t) = \bar{u}(x) + \tilde{u}(x,t)$

where $\frac{d^2\bar{u}(x)}{dx^2} = 0$ with $\bar{u}(0) = u_1, \bar{u}(l) = u_2$

$\Rightarrow \bar{u}(x) = u_1 + (u_2 - u_1)\frac{x}{l}$

and $\tilde{u}_t = \alpha\tilde{u}_{xx}$

with $I.C.: \tilde{u}(x,0) = u(x,0) - \bar{u}(x) = u_0 - [u_1 + (u_2 - u_1)\frac{x}{l}]$

B.C's: $\tilde{u}(0,t) = u(0,t) - \bar{u}(0) = u_1 - u_1 = 0$

$\tilde{u}(l,t) = u(l,t) - \bar{u}(l) = u_2 - u_2 = 0$

i.e. $\tilde{u}$ is a sol. of homo. b.c's $\tilde{u}(0,t) = \tilde{u}(l,t) = 0$

$\Rightarrow$ a typical eigenvalue prob.

Sol.: of $\tilde{u}$:

From previous example

$\Rightarrow \lambda_n = \frac{n\pi}{l}, n = 1, 2, \ldots$ *and* $\phi_n(x) = \sin\lambda_n x$

$\Rightarrow \tilde{u}(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha\lambda_n{}^2 t} \sin \lambda_n x$

For I.C.:

$\tilde{u}(x,0) = u_0 - [u_1 + (u_2 - u_1)\frac{x}{l}] = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$

$\rightarrow \quad A_n = \frac{2}{l}\int_0^l \{u_0 - [u_1 + (u_2 - u_1)\frac{x}{l}]\} \sin \frac{n\pi x}{l} dx$

$= \frac{2}{n\pi}[u_0 - (-1)^n u_0 + (-1)^n u_2 - u_1]$

Thus

$u(x,t) = \bar{u}(x) + \tilde{u}(x,t)$

$= u_1 + (u_2 - u_2)\frac{x}{l} + \sum_{n-1}^{\infty} \frac{2}{n\pi}[(u_0 - u_1) + (-1)^n(u_2 - u_0)]e^{\frac{-\alpha n^2 \pi^2 t}{l^2}} \sin \frac{n\pi x}{l}$

Ex.:

$$u_t = \alpha u_{xx} + F(x,t)$$

$$\text{I.C.:} \quad u(x,0) = f(x)$$

$$\text{B.C's:} \quad u(0,t) = u(l,t) = 0$$

Sol:

The eigenfuc. of homo. eq. together the homo. $b.c$'s be

$\phi_n(x) = \sin \frac{n\pi x}{l}$

Then we assume the solution in the expansion of the eigenfuc ϕ_n with coeffs. varying in time

$u(x,t) = \sum_{n=1}^{\infty} A_n(t)\phi_n(x)$

which satisfies the $b.c's$ identically.

Also, we expand F(x,t) in the form

$F(x,t) = \sum_{n=1}^{\infty} F_n(t)\phi_n(x)$

where $F_n(t) = \frac{2}{l}\int_0^l F(x,t) \sin \frac{n\pi x}{l} dx$

Then $Eq \Rightarrow$

$\sum_{n=1}^{\infty} \dot{A}_n(t)\phi_n(x) + \sum_{n=1}^{\infty} \alpha\lambda_n{}^2 A_n\phi_n(x) = \sum_{n=1}^{\infty} F_n\phi_n(x)$

$\Rightarrow \dot{A}_n(t) + \alpha\lambda_n{}^2 A_n(t) = F_n(t)$

with $I.C.$ $A_n(0) = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = A_n{}^o$

g.s. of $A_n(t)$

$$A_n(t) = e^{-\int \alpha\lambda_n{}^2 d\tau}[\int_0^t e^{\int \alpha\lambda_n{}^2 d\tau} F_n(\tau)d\tau + c]$$

By I.C. $A_n(0) = A_n{}^o \Rightarrow c = A_n{}^o$

therefore $A_n(t) = A_n{}^o e^{-\alpha\lambda_n{}^2 t} + \int_0^t e^{-\alpha\lambda_n^2(t-\tau)}F_n(\tau)d\tau$

Hence

u(x,t) = (Please write down the final form by yourself)

I-8 Solution of Inhomogeneous prob.

Consider a general inhomo. prob. for 1-d wave eq.

Eq.:$u_{tt} = c^2 u_{xx} + F(x,t) \quad 0 < x < l, t > 0$

$I.C's : u(x,0) = g(x)$

$u_t(x,0) = h(x) \qquad t > 0$

$B.C's : u(0,t) = f_1(t)$

$u(l,t) = f_2(t) \qquad 0 \le x \le l$

(A). Removal of Inhomo. B.C.'s

Set $u(x,t) = \bar{u}(x,t) + \tilde{u}(x,t)$

such that $\tilde{u}_{tt} = c^2 \tilde{u}_{tt} + \tilde{F}(x,t)$

$I.C/s : \tilde{u}(x,0) = \tilde{g}(x)$

$\tilde{u}_t(x,0) = \tilde{h}(x)$

$B.C's : \tilde{u}(0,t) = 0$

$\tilde{u}(l,t) = 0$

we shall solve this problem later.

case (1)

suppose $f_1(t) = A$

$f_2(t) = B \qquad A, B : const.$

i.e. $\bar{u}(0,t) = A \quad , \quad \bar{u}(l,t) = B$

$\Rightarrow \bar{u} = u(x,t) - \tilde{u}(x,t) = \bar{u}(x)$

so let us require $\bar{u}_{xx} = 0, \bar{u}(0) = A, \bar{u}(l) =$B

$\Rightarrow \bar{u}(x) = A + (B - A)\frac{x}{l}$

and this $\bar{u}(x,t)$ satisfies

$\tilde{u}_{tt} = c^2 \tilde{u}_{tt} + F(x,t)$

$I.C's : \tilde{u}(x,0) = g(x) - \bar{u}(x) = \tilde{g}(x)$

$\tilde{u_t}(x,0) = h(x) - 0 = \tilde{h}(x)$

$B.C's : \tilde{u}(0,t) = u(0,t) - \bar{u}(0) = A - A = 0$

$\tilde{u}(l,t) = u(l,t) - \bar{u}(l) = B - B = 0$

$\Rightarrow \tilde{u}(x,t)$ becomes a standard type of homo. *b.c's*

case (2)

If $u(0,t) = f_1(t)$, $u_t(0,t) = f_2(t)$ we require $\bar{u}(0,t) = f_1(t)$, $\bar{u}(l,t) = f_2(t)$ it is observed that $\bar{u}(x,t) = f_1(t)(1-\frac{x}{l}) + f_2(t)\frac{x}{l}$ will satisfy the b.c.'s

Hence eq. $\Rightarrow$

$$\bar{u}_{tt} + \tilde{u}_{tt} = c^2(\bar{u}_{xx} + \tilde{u}_{xx}) + F(x,t)$$

$$\Rightarrow f_{1tt}(1-\tfrac{x}{l}) + f_{2tt}\tfrac{x}{l} = c^2(\tilde{u}_{xx}) + F(x,t) - \tilde{u}_{tt}$$

$$\Rightarrow \tilde{u}_{tt} = c^2\tilde{u}_{xx} + F(x,t) - [f_{1tt}(1-\tfrac{x}{l}) - f_{2tt}\tfrac{x}{l}]$$

$$= c^2\tilde{u}_{xx} + \tilde{F}(x,t)$$

I.C's:

$$\tilde{u}(x,0) = u(x,0) - \bar{u}(x,0) = g(x) - f_1(0)(1-\frac{x}{l}) + f_2(0)\frac{x}{l}$$

$$= \tilde{g}(x)$$

$$\tilde{u}_t(x,0) = u_t(x,0) - \bar{u}_t(x,0) = h(x) - f_{1t}(0)(1-\frac{x}{l}) + f_{2t}(0)\frac{x}{l}$$

$$= \tilde{h}(x)$$

B.C.'s:

$$\tilde{u}(0,t) = u(0,t) - \bar{u}(0,t) = f_1(t) - f_1(t) = 0$$

$$\tilde{u}(l,t) = u(l,t) - \bar{u}(l,t) = f_2(t) - f_2(t) = 0$$

(B). Solution of inhomo. equation by eigenfuc. expansion. Consider an eigenvalue prob.

$$L[u] + \lambda r u = f(x) \quad a < x < b \quad (1)$$

+homo b.c's on [a,b]

Suppose we have found the eigenvalues and eigenfucs of the associated homo. prob.

$$L[v_i] + \lambda_i r v_i = 0 + \text{same homo. b.c's}$$

Then we may seek the solution of (1) by the method of eigenfuc. expansion, i.e. let

$u(x) = \sum_{i=1}^{\infty} c_i v_i(x)$

also expand $F(x) = \sum_{i=1}^{\infty} a_i v_i$ $\quad (F(x) = \frac{f}{r})$.

where $a_i = \frac{\int_a^b f v_i dx}{\int_a^b r {v_i}^2 dx}$

Eq. (1) $\Rightarrow$

$$L[\sum c_i v_i] + \lambda r \sum c_i v_i = f$$
$$-\sum c_i \lambda_i r v_i + \sum \lambda r c_i v_i = f$$
$$\sum c_i(\lambda - \lambda_i) v_i = \frac{f}{r} = F = \sum a_i v_i$$
$$\Rightarrow c_i = \frac{a_i}{\lambda - \lambda_i} \quad for \quad \lambda \neq \lambda_i$$

(a). If $\lambda \neq \lambda_i$, then

homo. $Lu + \lambda r u = 0$ has no sol.

inhomo. $Lu + \lambda r u = f$ has a unique sol.

$u(x) = \sum_{i=1}^{\infty} \frac{a_i}{\lambda - \lambda_i} v_i(x)$

(b). If λ is equal to one of the eigenvalue, say $\lambda = \lambda_j$, then

homo: $Lu + \lambda r u = 0$ has a solution $v_j(x)$

inhomo: $Lu + \lambda r u = f$ has

(i). Either no solution

(ii). or infinitly many sol.

$u(x) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda - \lambda_n} v_n(x) + c_j v_j$

where $n \neq j$, $c_j$is arb.

We now return back to the 1-d wave eq.

$$\begin{cases} EQ: \quad u_{tt} = c^2 u_{xx} + f(x,t) \\ I.C's: \quad u(x,0) = g(x) \\ u_t(x,0) = h(x) \\ B.C's \quad u(0,t) = u(l,t) = 0 \end{cases} \qquad (2)$$

Consider the associated homo. prob.

Eq.: $u_{tt} = c^2 u_{xx}$

I.C's: $u(x,0) = g_1(x)$

$u_t(x,0) = g_2(x)$

B.C's :$u(0,t) = u(l,t) = 0$

assume $u(x,t) = X(x)T(t)$

$$\frac{\ddot{T}}{c^2T} = \frac{X''}{X} = -\lambda^2 \quad (\lambda : real)$$

$X'' + \lambda^2 X = 0 + X(0) = X(l) = 0$

$\Rightarrow$ eigenvalues $\lambda_n = \frac{n\pi}{l}$ $n = 1, 2, ...$

eigenfucs. $X_n = \sin \lambda_n x$

$$\ddot{T} + c^2 {\lambda_n}^2 T = 0$$

$$T \sim \cos c\lambda_n t, \sin c\lambda_n t$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} (A_n \cos c\lambda_n t + B_n \sin c\lambda_n t) \sin \lambda_n x \quad \text{—(3)}$$

Apply $I.C's$

$$u(x,0) = g_1(x) = \sum_{n=1}^{\infty} A_n \sin \lambda_n x$$

$$\text{therefore} \quad A_n = \frac{2}{l} \int_0^l g_1(x) \sin \lambda_n x dx$$

$$u_t(x,0) = g_2(x) = \sum_{n=1}^{\infty} B_n c\lambda_n \sin \lambda_n x$$

$$\Rightarrow B_n = \frac{1}{c\lambda_n} \frac{2}{l} \int_0^l g_2(x) \sin \lambda_n x dx$$

From the sol. of homo. prob. represented by (3), we assume the sol. for (2) in the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \lambda_n x$$

which satisfies the b.c.'s identically.

we also expand

$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin \lambda_n x$

where $f_n(t) = \frac{2}{l}\int_0^l f(x,t)\sin\lambda_n x\,dx$

Then Eq $\Rightarrow$

$$\sum_{n=1}^{\infty} \ddot{T}_n(t)\sin\lambda_n x = -c^2\sum_{n=1}^{\infty}\lambda_n{}^2 T_n(t)\sin\lambda_n x + \sum_{n=1}^{\infty} f_n(t)\sin\lambda_n x$$

$$\Rightarrow \qquad \ddot{T}_n + c^2\lambda_n{}^2 T_n = f_n(t) \quad ---(4)$$

with the I.C.'s:

$$u(x,0) = g(x) = \sum_{n=1}^{\infty} T_n(0)\sin\lambda_n x$$

$$\Rightarrow T_n(0) = \frac{2}{l}\int_0^l g(x)\sin\lambda_n x\,dx \quad ---(5)$$

$$u_t(x,0) = h(x) = \sum_{n=1}^{\infty} \dot{T}_n(0)\sin\lambda_n x$$

$$\Rightarrow \dot{T}_n(0) = \frac{2}{l}\int_0^l h(x)\sin\lambda_n x\,dx \quad ---(6)$$

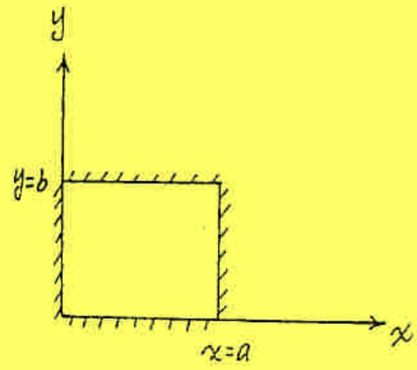
The g.s. of (4) $\Rightarrow$

$$T_n = \underbrace{A_n\cos c\lambda_n t + B_n\sin c\lambda_n t}_{T_{np}} + T_{np}$$

By using (5), & (6), we can det. A_n and B_n

I-9 Double Foruieṙ Series

Separation of var. in linear, eq. withe 2 indep. vars leads to an eigenvalue prob. governed by an o.d.e. When 3 indep. var. are present, sep. of vars leads to an eigenvalue prob. governed by a p.d.e. in z dims. Use of the sol. to a z-dim eigenvalue prob. in an eigenfuc. expansions requires a double Fourier series.



Consider

$$u_t = \alpha(u_{xx} + u_{yy})$$

$$= \alpha\nabla^2 u \qquad 0 < x < a, \quad 0 < y < b, \quad t > 0$$

$I.C: \quad u(x, y, 0) = u_0(x, y)$

B.C: all edges are insulted

$$u_x(0, y, t) = u_x(a, y, t) = 0$$

$$u_y(x, 0, t) = u_y(x, b, t) = 0$$

Assume $u(x, y, t) = T(t)\Omega(x, y)$

$$\Rightarrow \frac{\dot{T}}{\alpha T} = \frac{\nabla^2\Omega}{\Omega} = -\lambda^2 \quad (\lambda: \text{ real}) \quad --- (1)$$

$$\Rightarrow \dot{T} + \lambda^2\alpha T = 0 \quad T \sim e^{-\alpha\lambda^2 t}$$

$$\nabla^2\Omega + \lambda^2\Omega = 0$$

with $\Omega_x(0, y) = \Omega_x(a, y) = \Omega_y(x, 0) = \Omega_y(x, l) = 0$ we can prove the eigenfucs $\Omega_n(x, y)$ ar orthogonal (EXERCISE)

i.e. let $\nabla^2\Omega_m + \lambda_m^2\Omega_m = 0$

$\nabla^2\Omega_n + \lambda_n^2\Omega_n = 0 \quad$ + homo b.c's

$\int_0^a \int_0^b \Omega_m(x, y)\Omega_n(x, y)dxdy = 0 \quad for \quad m \neq n$

Solution of Ω:

Let $\Omega(x, y) = X(x)\bar{Y}(y)$

$(1) \Rightarrow \frac{X''}{X} = -\lambda^2 - \frac{\bar{Y}''}{\bar{Y}} = -\beta^2 (\beta : real)$

$$\Rightarrow \begin{cases} X'' + \beta^2 X = 0 \\ X'(0) = X'(a) = 0 \end{cases}$$

$$\bar{Y}'' + (\lambda^2 - \beta^2)\bar{Y} = 0$$

$$\bar{Y}'(0) = \bar{Y}'(b) = 0$$

$\left(\text{or}\bar{Y}'' + \gamma^2\bar{Y} = 0\right)$ where $\lambda^2 = \beta^2 + \gamma^2$

It's easy to find

$\beta_m = \frac{m\pi}{a}$

$X_m(x) = \cos\frac{m\pi x}{a}$

$\gamma_n = \frac{n\pi}{b}$

$\bar{Y}_n(y) = \cos\frac{n\pi y}{b}$

and thus

$$\lambda_{mn}{}^2 = \beta_m^2 + \gamma_n^2 = (\tfrac{m\pi}{a})^2 + (\tfrac{n\pi}{b})^2$$

and

$$\Omega_{mn}(x,y) = \cos\tfrac{m\pi x}{a}\cos\tfrac{n\pi y}{b}$$

so the general sol.

$$u(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}e^{-\alpha\lambda_{mn}{}^2 t}\cos\tfrac{m\pi x}{a}\cos\tfrac{n\pi y}{b}$$

Apply I.C.

$$u_0(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}\cos\tfrac{m\pi x}{a}\cos\tfrac{n\pi y}{b}$$

This is to represent $u_0(x,y)$ in a double Fouries expansion.

$$\Rightarrow A_{mn} = \tfrac{1}{N_{mn}}\int_0^a\int_0^b u_0(x,y)\cos\tfrac{m\pi x}{a}\cos\tfrac{n\pi x}{b}dxdy$$

and

$$N_{mn} = \int\int \Omega_{mn}{}^2(x\cdot y)dxdy$$

$$= \int_0^a\int_0^b \cos^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b}dxdy$$

$$= \begin{cases} ab & \text{for n=m=0} \\ \frac{ab}{2} & \text{for n=0 or m=0} \\ \frac{ab}{4} & \text{for m,n} > 0 \end{cases}$$

I-10 Fourier-Bessel Series

Bessel's eq.: frequently occurs in probs of mathematical physics involving cylindrical geometry.

(A)Bessel. eq.

$$x^2y'' + xy' + (\lambda^2x^2 - p^2)y = 0 \quad ---(1)$$

We can find a power series sol. of(1) by the method of Frobenious. The general solution of (1) is

$$y(x) = \begin{cases} c_1J_p(\lambda x) + c_2J_{-p}(\lambda x) & \text{for } p \neq 0, 1, 2, ... \\ c_1J_p(\lambda x) + c_2Y_p(\lambda x) & \text{for all values of p} \end{cases}$$

J_p: Bessel's func. of the first kind of order p.

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{x}{2})^{2k+p}}{k(k+p)!}$$

$$J_{-p}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{x}{2})^{2k-p}}{k(k-p)!}$$

Y_p : Bessel's fuc. of the 2nd kind of order p

$$\begin{cases} Y_p(x) = \frac{2}{\pi}[(\ln(\frac{x}{2}) + 0.5772159)J_p(x) - (\frac{1}{\pi})\sum_{k=0}^{p-1} \frac{(p-k-1)!(\frac{x}{2})^{2k+p}}{k!} \\ +\frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^{k+1} \times (h_p + h_{k+p})\frac{(\frac{x}{2})^{2k+p}}{k!(k+p)}, \quad \text{when} \quad p = 1, 2, ... \\ Y_o(x) = \frac{2}{\pi}[J_o(x)(\ln(\frac{x}{2}) + 0.5772159) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}hm}{2^{2m}(m!)^2}x^{2m} \\ Y_p(x) = \frac{\cos p\pi J_p(x) - J_{-p}(x)}{\sin p\pi} \\ p \neq 0, 1, 2, 3, ...\text{where} \quad h_k = 1 + \frac{1}{2} + ... + \frac{1}{k} \end{cases}$$

It is noted that $J_{-p}, Y_p, \rightarrow -\infty \quad as \quad x \rightarrow 0$

(B)Modified Bessel Equation

$$x^2y'' + xy\prime - (\lambda^2x^2 + p^2)y = 0 \quad ---(2)$$

The general sol. of (2)

$$y(x) = \begin{cases} c_1I_p(\lambda x) + c_2K_p(\lambda x) & \text{for all p} \\ c_1I_p(\lambda x) + c_2I_{-p}(\lambda x) & \text{for p} \neq 0, 1, 2, ... \end{cases}$$

Where $I_p(x)$: modified Bessel fuc of the 1st kind of order p

$$= \sum_{k=0}^{\infty} \frac{(\frac{x}{2})^{2k+p}}{k!(k+p)!}$$

K_p (x) = modified B.f. of the 2nd kind of order p.

$K_p(x) = \frac{\pi}{2} i^{p+1} H_p^{(1)}(ix)$ where p=0,1,2,3

$= \frac{\pi}{2} \frac{I_{-p}(x) - I_p(x)}{\sin p\pi}$ when $p \neq 0, 1, 2, 3$

$H_p(x) = J_p(ix) + iY_p(ix)$: 1st Hankel fuc. of order p.

(C)Eigenfuc Expansion

Consider a S-L system

$EQ : (xu')' + (\lambda^2 x - \frac{p^2}{x} u) = 0 \quad a < x < b \quad ------(3)$

with homo. b.c's

$u(a) + \alpha u'(a) = 0$

$u(b) + \beta u'(b) = 0$

The g.s. of (3)

$u(x) = \mathbf{Z}_p(\lambda x) = \mathbf{c_1 J}_p(\lambda x) + \mathbf{c_2 Y}_p(\lambda x)$

applying the b.c's, we have two homo. alegebraic eqs. for the unknowns c_1, c_2., then from the condition of nontrivial sol. we have the so-called char. eq. to determine the eigenvalues $\lambda_n, \quad n = 1, 2, ...$

The char. eq. is geneally a transcendental eq. the eigenvalue should be found from tables or by the graphical method. Corresp. to eigenvalue λ_n, we find the eigenfuc $\mathbf{Z}_p(\lambda_n x)$, which form a complete orthogonal set (by $S - L$ theorem)

So we can express any piecewise continuous fuc. $f(x)$ on $L_2[a, b]$ in terms of $\mathbf{Z}_p(\lambda_n x)$

i.e. $f(x) = \sum_{i=1}^{\infty} \mathbf{A}_p \mathbf{Z}_p(\lambda_i x)$

The Orthogonality Condition

$$\int_a^b x\mathbf{Z}_p(\lambda_m x)\mathbf{Z}_p(\lambda_n x)dx = \begin{cases} 0 & \text{when } m \neq n \\ c_n & \text{when } m=n \end{cases}$$

$$\Rightarrow A_n = \frac{\int_a^b xf(x)\mathbf{Z}_p(\lambda_n x)dx}{c_n}$$

where $c_n = \int_a^b x{\mathbf{Z}_p}^2(\lambda_n x)dx$

Let $u = \mathbf{Z}_p(\lambda_m x), v = \mathbf{Z}_p(\lambda_n x)$ be the eigenfucs corresp. to the distinct eigenvalues λ_m, λ_n respectively

i.e. $u'' + \frac{u'}{x} + ({\lambda_m}^2 - \frac{p^2}{x^2})u = 0 \quad ---(4)$

$v'' + \frac{v'}{x} + (\lambda_n{}^2 - \frac{p^2}{x^2})v = 0 \quad - - - (5)$

$\int_a^b [(4)\text{x}xv - (5)\text{x}xu]dx \Rightarrow$

$\int_a^b [x(u''v - uv'') + (u'v - uv')]dx + (\lambda_m{}^2 - \lambda_n{}^2)\int_a^b xuvdx = 0$

$\Rightarrow \int_a^b \frac{d}{dx}[x(u'v - uv')]dx = (\lambda_n{}^2 - \lambda_m{}^2)\int_a^b xuvdx$

or $(\lambda_n{}^2 - \lambda_m{}^2)\int_a^b xuvdx = [x(u'v - uv')]|_a^b$

From the homo. $b.c./$s

$\Rightarrow (\lambda_n{}^2 - \lambda_m{}^2)\int_a^b xuvdx = 0$

Since $\lambda_n \neq \lambda_m$

$\Rightarrow \int_a^b x\text{Z}_p(\lambda_m x)\text{Z}_p(\lambda_n x)dx = 0$

* How to complete $c_n = \int_0^l xJ_p^2(\lambda_n x)dx = ?$

Let u$=J_p(\lambda_n x)$

From $(x\frac{du}{dx})' + (\lambda_n{}^2 x - \frac{p^2}{x})u = 0 \qquad (6)$

$\int_0^l (6)\text{x}(2x\frac{du}{dx})dx \Rightarrow$

$\int_0^l (\lambda_n{}^2 x^2 - p^2)\frac{d}{dx}(u^2)dx = -\int_0^l \frac{d}{dx}[(x\frac{du}{dx})^2]dx$

$\Rightarrow [(\lambda_n{}^2 x^2 - p^2)(u^2)]_0^l - 2\lambda_n{}^2\int_0^l xu^2 dx = -[x^2(\frac{du}{dx})^2]_0^l$

since $J_p(0) = 0$

$\rightarrow \int_0^l xJ_p{}^2(\lambda_n x)dx$

$= \frac{1}{2\lambda_n{}^2}\{(\lambda_n{}^2 l^2 - p^2)J_p{}^2(\lambda_n l) + l^2[J_p{}'(\lambda_n l)]^2\}$

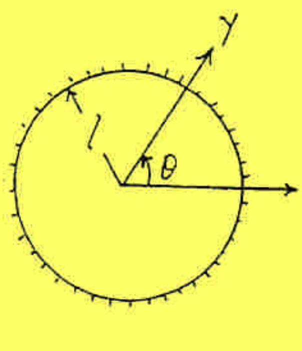
now $J_p{}' = -\lambda_n J_{p+1} + \frac{p}{l}J_p$

(i). so if the b.c at $x = l$ such that $J_p(\lambda_n l) = 0$ then $c_n = \frac{l^2}{2}[J_{p+1}(\lambda_n l)]^2$

(ii). if $J_p{}'(\lambda_n l) = 0$ then $c_n = \frac{\lambda_n{}^2 l^2 - p^2}{2\lambda_n{}^2}[J_p{}^2(\lambda_n l)]$

Ex.: Vibraton of Circular Membrane

$u_{tt} - c^2(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$



$I.C's : u(r,\theta,0) = f(r,\theta)$

$u_t(r,\theta,0) = 0$

$B.C's : u(1,\theta,t) = 0$

Sol.:

Assume $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$

$$\Rightarrow \frac{\ddot{T}}{c^2 T} = \frac{1}{R}(R'' + \frac{1}{r}R') + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda^2 \quad (\lambda : real)$$

and

$$\frac{1}{R}(r^2 R'' + rR') + \lambda^2 r^2 = -\frac{\Theta''}{\Theta} = \mu^2 \quad (\mu : real)$$

we thus have the eqs.:

$\ddot{T} + c^2\lambda^2 T = 0$

$\Theta'' + \mu^2\Theta = 0$

$r^2 R'' + rR' + (\lambda^2 r^2 - \mu^2)R = 0$

Sol.: of Θ :

$\Theta = c_1 \cos\mu\theta + c_2 \sin\mu\theta$

since it is periodic in circular dir. for single-valued sol.

$\Theta[\mu\theta] = \Theta[\mu(2\pi + \theta)]$

$\rightarrow \mu_n = n = 0, 1, 2...$

$\Rightarrow \Theta_n(\theta) = c_n \cos n\theta + d_n \sin n\theta$

Sol. of R:

$r^2 R_n{}'' + rR_n{}' + (\lambda_n{}^2 r^2 - n^2)R_n = 0$

$\rightarrow R_n(r) = a_n J_n(\lambda_n r) + b_n Y_n(\lambda_n r)$

since $Y_n(0) = -\infty \quad \Rightarrow b_n = 0$ and from $R_n(1) = a_n J_n(\lambda_n) = 0$ gives the eigenvalues $\lambda_n j$ such that $z_j = \lambda_{nj}$ is the jth root of $J_n(x) = 0$

so $\ddot{T}_{nj} + c^2\lambda_{nj}{}^2 T = 0 \quad with \quad \dot{T}_{nj}(0) = 0$

$\Rightarrow T_{nj} = \cos c\lambda_{nj}t$

Hence the general sol. is

$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} J_n(\lambda_{nj}r)(A_{nj}\cos n\theta + B_{nj}\sin n\theta)\cos c\lambda_{nj}t$

For rhw I.C $\Rightarrow$

$u(r,\theta,0) = f(r,\theta) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} J_n(\lambda_{nj}r)(A_{nj}\cos n\theta + B_{nj}\sin n\theta)$

This is called the Fourier-Bessel expansion of $f(r,\theta)$ first, find

$\sum_{j=1}^{\infty} \begin{Bmatrix} A_{nj} \\ B_{nj} \end{Bmatrix} J_n(\lambda_{nj} r) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(r,\theta) \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} d\theta = \begin{Bmatrix} g_c(r) \\ g_s(r) \end{Bmatrix}$

then can find

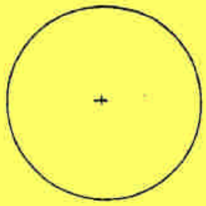
$$\begin{Bmatrix} A_{nj} \\ B_{nj} \end{Bmatrix} = \frac{\int_0^1 r \begin{Bmatrix} g_c(r) \\ g_s(r) \end{Bmatrix} J_n(\lambda_n r) dr}{\int_0^1 r {J_n}^2(\lambda_{nj} r) dr}$$

$$= \frac{2}{{J_{p+1}}^2(\lambda_n)} \int_0^1 r \begin{Bmatrix} g_c(r) \\ g_s(r) \end{Bmatrix} J_n(\lambda_{nj} r) dr$$

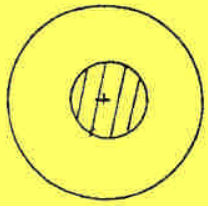
Mode of Vibration

(i)Symmetric mode (n=0)

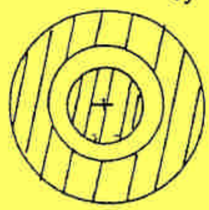
The value of λ_{0j} are obtained from $J_0(\lambda_{0j}) = 0 \quad \Rightarrow \quad \lambda_{0j} = 2.405, 5.520, 8.654$



$J_0(\lambda_{01} r)$..



$J_0(\lambda_{02} r)$..

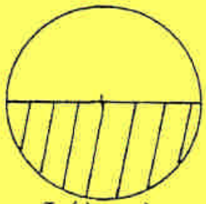


$J_0(\lambda_{03} r)$..

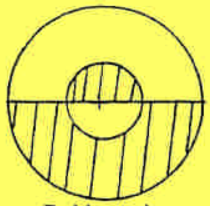
(ii)Unsymmetric mode (n=1)

$J_1(\lambda_{1j}) = 0$

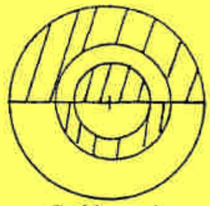
$\Rightarrow \lambda_{1j} = \quad 3.832, 7.016, 10.173$



$J_1(\lambda_{11} r)$..



$J_1(\lambda_{12} r)$..



$J_1(\lambda_{13} r)$..

Ex.:

$u_{tt} - c^2(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}) = F(r,\theta)\cos\omega t$ (Forced vibration of a circular membrance).

$I.C's : u(r,\theta,0) = u_t(r,\theta,0) = 0$

$B.C: \quad u(1,\theta,t) = 0$

Sol.:

The eigenfucs of the associated homo. prob. are found to be

$J_n(\lambda_{nj} r) \begin{Bmatrix} cosn\theta \\ \sin n\theta \end{Bmatrix}$

So we may seek the sol. of the inhomo. prob. in the form

$u(r,\theta,t)=\sum_{n=0}^{\infty}\sum_{j=1}^{\infty} J_n(\lambda_{nj}r)[A_{nj}(t)\cos n\theta + B_{nj}(t)\sin n\theta]$

and expand $F(r,\theta)$

$=\sum_{n=0}^{\infty}\sum_{j=1}^{\infty} J_n(\lambda_{nj}r)[a_{nj}\cos n\theta + b_{nj}\sin n\theta]$

we can find a_{nj} *and* b_{nj}

Then EQ $\Rightarrow$

$$\sum_{n=0}^{\infty}\sum_{j=1}^{\infty} J_n(\lambda_{nj}r)[\ddot{A}_{nj}(t)\cos n\theta + \ddot{B}_{nj}(t)\sin n\theta]$$

$$-c^2\sum_{n=0}^{\infty}\sum_{j=1}^{\infty}(-\lambda_{nj})^2 J_n(\lambda_{nj}r)(A_{nj}\cos n\theta + B_{nj}\sin n\theta)$$

$$=\cos\omega t\sum_{n=0}^{\infty}\sum_{j=1}^{\infty} J_n(\lambda_{nj}r)(a_{nj}\cos n\theta + b_{nj}\sin n\theta)$$

$$\Rightarrow \ddot{A}_{nj}(t) + c^2{\lambda_{nj}}^2 A_{nj}(t) = a_{nj}\cos\omega t$$

$$\ddot{B}_{nj}(t) + c^2{\lambda_{nj}}^2 B_{nj}(t) = b_{nj}\cos.\omega t$$

with I.C's

$A_{nj}(0)=\dot{A}_{nj}(0)=0$

$B_{nj}(0)=\dot{B}_{nj}(0)=0$

$\Rightarrow \left\{ {A_{nj}(t) \atop B_{nj}(t)} \right\} = \frac{1}{{\lambda_{nj}}^2c^2-\omega^2}\left\{ {a_{nj} \atop b_{nj}} \right\}[\cos\omega t - \cos(c\lambda_{nj}t)]$

I-11 Legendre Series

(A)Legendre Eq.:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad n = 0, 1, 2, ...$$

The g.s.

$$y(x) = c_1 P_n(x) + c_2 Q_n(x) \quad \text{for} \quad |x| < 1$$

where

$$P_n(x) = \text{nth L. polynomial of the 1st kind}$$

$$= \sum_{k=0}^{N} \frac{(-1)^k (2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}$$

where

$$N = \begin{cases} n/2 & \text{when} \quad \text{n:even} \\ (n-1)/2 & \text{when} \quad \text{n:odd} \end{cases}$$

Or,

$$P_n(x) = \begin{cases} \frac{u_n(x)}{u_n(1)} & \text{when n is even} \\ \frac{v_n(x)}{v_n(1)} & \text{when n is odd} \end{cases}$$

where

$$U_n(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \frac{n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!}x^6 + ...$$

$$V_n(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \frac{(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!}x^7 + ...$$

$Q_n(x)$: nth Legendrel polynomial of the 2nd kind

$$= \begin{cases} u_n(1)v_n(x) & \text{when } n \text{ is even} \\ -v_n(1)u_n(x) & \text{when } n \text{ is odd} \end{cases}$$

Note that $P_n(x)$ is finite everywhere in $[-1, 1]$ but $Q_n(x)$ is not finite at the end pts. $x = \pm 1$.

$P_n(x)$ may also be expressed as Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

The Orthogonality Relation:since

$$[(1-x^2)P_n{}'(x)]' + n(n+1)P_n(x) = 0 \quad ---(1)$$

$$[(1-x^2)P_m{}'(x)]' + m(m+1)P_m(x) = 0 \quad ---(2)$$

$$\int_{-1}^{1} [(1)\text{x}P_m(x) - (2)\text{x}P_n(x)]dx \Rightarrow$$

$$\int_{-1}^{1}[P_n(x)\{(1-x^2)P_m{}'\}' - P_m(x)\{(1-x^2)P_n(x)'\}']dx$$
$$= (n-m)(n+m+1)\int_{-1}^{1} P_m(x)P_n(x)dx$$

On integrating by parts

$$\Rightarrow \int_{-1}^{1} P_m(x)P_n(x)dx = 0 \quad (if \quad n \neq m)$$

How to complete $\int_{-1}^{1} P_n(x)^2 dx = ?$

Denote $I_{mn} = \int_{-1}^{1} P_m(x)P_n(x)dx$ Without loss of generality, we let m$\leq n$, and by using of Rodrigues formula, we have

$$I_{mn} = \frac{1}{2^{m+n}m!n!}\int_{-1}^{1}[(\frac{d}{dx})^m(x^2-1)^m][(\frac{d}{dx})^n(x^2-1)^n]dx$$

On integrating by parts, $\Rightarrow$

$$\frac{1}{2^{m+n}m!n!}[\{(\frac{d}{dx})^m(x^2-1)^m\}\{(\frac{d}{dx})^{n-1}(x^2-1)^n\}]_{-1}^{1}$$
$$-\frac{1}{2^{m+n}m!n!}\int_{-1}^{1}[(\frac{d}{dx})^{m+1}(x^2-1)^m][(\frac{d}{dx})^{n-1}(x^2-1)^n]dx$$

=........ (n times)

$$= \frac{(-1)^n}{2^{m+n}m!n!}\int_{-1}^{1}[(\frac{d}{dx})^{m+n}(x^2-1)^m](x^2-1)^n dx$$

If $m < n$, then $(\frac{d}{dx})^{m+n}(x^2-1)^m = 0$

$$\rightarrow I_{mn} = 0 \quad \text{for} \quad m < n$$

If $m = n$ then $(\frac{d}{dx})^{m+n}(x^2-1)^m = (2m)! = (2n)!$

$$I_{nn} = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n dx$$
$$= \frac{(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(1-x^2)^n dx$$

let x=2u-1 $\Rightarrow$

$$I_{nn} = \frac{2(2n)!}{(n!)^2}\int_{0}^{1} u^n(1-u)^n du$$
$$= \frac{2(2n)!}{(n!)^2}\beta(n+1, n+1)$$

where $\beta(r,s) is the Beta fuc., defined by$

$$\beta(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} = \int_{0}^{1} x^{r-1}(1-x)^{s-1}dx$$
$$\Rightarrow I_{nn} = \frac{2(2n)!}{(n!)^2}\cdot\frac{n!n!}{(2n+1)!} = \frac{2}{2n+1}$$
$$\Rightarrow \int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2m+1}\delta_{mn}$$

(B)Associated Legendre Eq.:

$$(1-x^2)y'' - 2xy\prime + [n(n+1) - \frac{m^2}{1-x^2}]y = 0$$

The g.s is

$$y(x) = c_1 P_n{}^m(x) + c_2 Q_n{}^m(x) \quad \text{for} \quad |x| < 1$$

where $P_n{}^m(x), Q_n{}^m(x)$ are called the associated Legendre fucs. of degree n, and order m, of the 1st kind and 2nd kind respectively, and are

$$P_n{}^m(x) = (1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m} \quad (EXERCISE)$$

$$Q_n{}^m(x) = (1-x^2)^{m/2} \frac{d^m Q_n(x)}{dx^m}$$

Note that $P_n{}^m(x)$ is finite everywhere in [-1,1] $Q_n{}^m(x)$ is not finite at $x \pm 1$

The Orthogonality Relation

$$\int_{-1}^{1} P_n{}^m(x) P_k{}^m(x) = \frac{2(n+m)!}{(2n+1)(n-m)!}{}_{nk} \quad (EXERCISE)$$

(C)Expansion Theorem

Physically, we are often interested in the case $x \in [-1,1]$ hence in such case only the eigenfuc. $P_n(x)$, *or* $P_n^m(x)$ is convergent expansion.

$$f(x) = \sum_{n=0}^{\infty} C_n P_n(x)$$

where

$$C_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx$$

or

$$f(x) = \sim_{n=0}^{\infty} d_n P_n^m(x)$$

where

$$d_n = \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \int_{-1}^{1} f(x) P_n^m(x) dx$$

(D)Surface Harmonics

Consider the Laplace eq. in spherical coords

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial u}{\partial r}) + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial \varphi}(\sin\varphi \frac{\partial u}{\partial \varphi}) + \frac{1}{r^2 \sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Look for sol. $u(r,\varphi,\theta) = R(r)H(\varphi)M(\theta)$

We have the following separated eqs.

$$\frac{d^2M}{d\theta^2} + m^2M = 0$$

$$\frac{1}{\sin\varphi}\frac{d}{d\varphi}(\sin\varphi\frac{dH}{d\varphi}) - \frac{m^2H}{\sin^2\varphi} + \lambda H = 0$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dR}{dr}) - \frac{\lambda}{r^2}R = 0$$

where m^2, λ are the separation consts.

Sol. of $M(\theta)$

$$M_m(\theta) = a_m \cos m\theta + b_m \sin m\theta, \quad m = 0, 1, 2, ...$$

Sol. of $H(\varphi)$ *let* $x = \cos\varphi$

$\Rightarrow \quad (1-x^2)\frac{d^2H}{dx^2} - 2x\frac{dH}{dx} + (\lambda - \frac{m^2}{1-x^2})H = 0$

This eq. can be shown that the sol. exists only for $\lambda = n(n+1), \quad n = 0, 1, 2, ...$

$\Rightarrow (1-x^2)\frac{d^2H}{dx^2} - 2x\frac{dH}{dx} + (n(n+1) - \frac{m^2}{1-x^2})H = 0$

associated L.eq.

g.s. $\Rightarrow$

$$H(\varphi) = c_1{P_n}^m(\cos\varphi) + c_2{Q_n}^m(\cos\varphi)$$

However, the argument $-1 \leq \cos\varphi \leq 1$ but ${Q_n}^m(\pm 1)$ is unbounded.

$\Rightarrow H(\varphi) \sim {P_n}^m(\cos\varphi)$

Sol. of $R(r) : \frac{d}{dr}(r^2\frac{dR}{dr}) - n(n+1)R = 0$

$\Rightarrow R(r) \sim r^n, r^{-(n+1)}$

Define the surface harmonics

$${C_n}^m(\varphi, \theta) = {P_n}^m(\cos\varphi)\cos m\theta$$

$${S_n}^m(\varphi, \theta) = {P_n}^m(\cos\varphi)\sin m\theta$$

Then the g.s. of L. eq. is

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty}\Big\{r^n \sum_{m=0}^{\infty}[a_{nm}{C_n}^m(\varphi, \theta) + b_{nm}{S_n}^m(\varphi, \theta)]$$

$$+r^{-(n+1)} \sum_{m=0}^{\infty} [c_{nm} C_n{}^m(\varphi,\theta) + d_{nm} S_n{}^m(\varphi,\theta)] \Big\}$$

The Orthogonality Relation.

Let S_R denotes the surface of a sphere of radious R with center at origin, then $\int_{S_R} C_n{}^m S_{n'}{}^{m'} d\sigma = 0$ for all m, n, m', n'

and

$$\frac{1}{4\pi R^2} \int_{S_R} \begin{Bmatrix} C_n{}^m C_{n'}{}^{m'} \\ S_n{}^m S_{n'}{}^{m'} \end{Bmatrix} d\sigma = \delta_{mm'} \delta_{mn'} \frac{(n+m)!}{2(2n+1)(n-m)!}$$

(EXERCISE)

The Expansion Theorem

if any fuc. $f(\varphi,\theta)$ is well behaved in $0 \le \varphi \le \pi, \quad 0 \le \theta \le 2\pi,$ then f has a convergent expansion.

$f(\theta,\varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{nm} C_n{}^m(\varphi,\theta) + b_{nm} S_n{}^m(\varphi,\theta)$

where

$$\begin{Bmatrix} a_{nm} \\ b_{nm} \end{Bmatrix} = \frac{1}{d_{nm}} \int_{S_R} f(\varphi,\theta) \begin{Bmatrix} C_n{}^m(\varphi,\theta) \\ S_n{}^m(\varphi,\theta) \end{Bmatrix} d\sigma$$

$$d_{nm} = \pi R^2 \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}$$

(E)Spherical Harmonics

For combining the angular factors, and constructing orthogonal fucs. over the unit sphere, we will call

$$Y_{lm}(\Omega) = \left[\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}\right]^{\frac{1}{2}} P_c^{|m|}(\cos\varphi) e^{im\theta} \quad \begin{cases} (\text{-}1)^m & \text{m} \ge 0 \\ 1 & m < 0 \end{cases}$$

the spherical harmonics. we can verify that

$$Y_{lm}(\Omega) = \frac{1}{2^l l!} \left[\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}\right]^{\frac{1}{2}} e^{im\theta} (-\sin\varphi)^m \frac{d^{l+m}}{d(\cos\varphi)^{l+m}} (\cos^2\varphi - 1)^l (EXERCISE)$$

It follows that $Y_{l,-m}(\Omega)d\Omega = (-1)^m Y^*_{lm}(\Omega)$ where Y_{lm} is chosen for the normalization such that

$$\int Y^*_{lm}(\Omega)Y_{l'm'}(\Omega)d\Omega = \delta_{ll'}\delta_{mm'}$$

Expansion Theorem

$$f(\Omega) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}Y_{lm}(\Omega)$$

where

$$A_{lm} = \int f(\Omega)Y^*_{lm}(\Omega)d\Omega$$

Ex.

For a concentric spherical perfect fluid in between. If hold sphere $r = b$ fixed, oscillate sphere $r = b$ with small amplitude along $z-$axial.

$$EQ.: \quad \nabla^2\phi = 0 \quad a < r < b$$

$$B.C's: \frac{\partial\phi}{\partial r}|_{r=b} = 0$$

If the motion of the sphere $r = a$ is

$$\underline{V_s} = A\cos\omega t\underline{k}$$

then the b.c for the inner sphere (no-slip) is $\underline{n}\cdot(\nabla\phi - \underline{V_s})|$ inner sphere $= 0$

If A is very small, we can approx. this b.c$'s$ as

$$\underline{n}\cdot(\nabla\phi - \underline{V_s})|_{r=a} = 0$$

Sol.:

Since the flow is around z-axis, the flow is axisymmetric (i.e. indep. of θ)

$\Rightarrow$ look for sol. of ϕ in the form

$$\phi = \sum_{n=0}^{\infty}[c_n r^n + d_n r^{-(n+1)}]P_n(\cos\varphi)$$

at $r = b \Rightarrow$

$$\frac{\partial\phi}{\partial r}|_{r=b} = \int_{n=0}^{\infty}[nc_n b^{n-1} - (n+1)d_n b^{-(n+2)}]P_n(\cos\varphi) = 0 \quad \text{for all} \quad \varphi$$

$$\Rightarrow nc_n b^{n-1} = (n+1)d_n b^{-(n+2)} ------(1)$$

at $r = a$,

$$\int_{n=0}^{\infty}[nc_n a^{n-1} - (n+1)d_n a^{-(n+2)}]P_n(\cos\varphi) = A\cos\omega t\cos\varphi$$

since $P_0(\cos\varphi) = 1, \quad P_1(\cos\varphi) = \cos\varphi$

$\rightarrow$

$d_0 a^{-2} + (c_1 - 2d_1 a^{-3} - A\cos\omega t)P_1(\cos\varphi)$

$+ \int_{n=0}^{\infty}[nc_n a^{n-1} - (n+1)d_n a^{-(n+2)}]P_n(\cos\varphi) = 0$

$\Rightarrow d_0 = 0$

$c_1 - 2d_1 a^{-3} - A\cos\omega t = 0 - - - - - (2)$

$nc_n a^{n-1} - (n+1)d_n a^{-(n+2)} = 0 \quad for \quad n \geq 2 - - - - - (3)$

let $n = 0$ in (1) $\Rightarrow c_0$ is arb.,if $c_0 \neq 0$, it gives a const. only (no contribution to velocity field) $\Rightarrow$ let $c_n = 0$

let $n = 1$ in (1) $\quad \Rightarrow c_1 = 2d_1 b^{-3}$

then (2) $\quad \Rightarrow d_1 = -\frac{Aa^3 b^3 \cos\omega t}{2(b^3 - a^3)}$

and then $\quad c_1 = -\frac{Aa^3 \cos\omega t}{(b^3 - a^3)}$

and from (1),& (3) we find

$c_n = d_n = 0 \quad$ for $\quad n \geq 2$

$\Rightarrow \phi = -\frac{Aa^3}{2(b^3 - a^3)} \cos\phi(2r + \frac{b^2}{r^2})\cos\omega t$

motions are in phase with the inner sphere.

I-12 Fourth Order Equation

Consider the transverse vibration of an elastic beam.

$$v_{tt} + c^2 v_{xxxx} = \frac{q}{\rho} \qquad 0 < x < l, \quad t > 0$$

where v: transverse displacement, $c^2 = \sqrt{\frac{EI}{\rho}}$, EI: rigidity, ρ:density, $\frac{q}{\rho}$:loading intensity

For the free vibration, q=0

$$v_{tt} + c^2 v_{xxxx} = 0 - - - - - - (1)$$

B.C$'s$: For simply supported ends

$$v(0,t) = v_{xx}(0,t) = 0$$

$$v(l,t) = v_{xx}(l,t) = 0$$

I.C$'s$: given initial displacement and velocity

$$v(x,0) = f(x)$$

$$v_t(x,0) = g(x)$$

Sol.: let $v(x,t) = X(x)T(t)$

$$(1) \Rightarrow \frac{\ddot{T}}{c^2 T} = -\frac{X''''}{X} = -\lambda^4 \quad (\lambda^4 : (+))$$

$$\ddot{T} + c^2\lambda^4 T = 0 - - - - - - (2)$$

$$X'''' - \lambda^4 X = 0 - - - - - - (3)$$

Sol. of $X(x)$:

$$X(0) = X''(0) = X(l) = X''(l) = 0$$

By the method, we have the g.s. for (3) $\Rightarrow$

$$\underline{X}(x) = c_1(\cos\lambda x + \cosh\lambda x) + \frac{c_2}{\lambda}(\sin\lambda x + \sinh\lambda x)$$

$$+\frac{c_3}{\lambda^2}(\cos\lambda x - \cosh\lambda x) + \frac{c_4}{\lambda^3}(\sin\lambda x - \sinh\lambda x) - - - - - - (4)$$

By substituting the $b.c's$, we can find the eigenvalues λ^4.

$*$ Properties of the eigenvalue λ^4

If ${\lambda_m}^4$ & ${\lambda_n}^4$ are two distinct eigenvalues,i.e.:

$$X_m'''' = {\lambda_m}^4 X_m - - - - - - (5)$$

$$X_n'''' = {\lambda_n}^4 X_m - - - - - - (6)$$

$(5) \Rightarrow$

$$\int_0^l X_m X_m'''' dx = {\lambda_m}^4 \int_0^l {X_m}^2 dx = [X_m X_m''' - {X_m}' {X_m}'']|_0^l + \int_0^l ({X_m}'')^2 dx$$

i.e. $\lambda_m{}^4 \int_0^l X_m{}^2 dx = \int_0^l (X_m'')^2 dx$

$\Rightarrow \lambda_m{}^4 > 0$ for all m unless $X_m'' = 0$ for some value of m.

If $X_m'' = 0$ then $\lambda_m = 0$, with $\lambda_0 = 0$

$(4) \Rightarrow X_0(x) = 2c_1 + 2c_2 x - c_3 x^2 - \frac{1}{3} c_4 x^3$

For the b.c$'s$ $\Rightarrow$

$X_0(0) = 0 \Rightarrow c_1 = 0$

$X_0''(0) = 0 \Rightarrow c_3 = 0$

$X_0''(l) = 0 \Rightarrow c_4 l = 0 \Rightarrow c_4 = 0$

$X_0(l) = 0 \Rightarrow c_2 l = 0 \Rightarrow c_2 = 0$

Thus only trivial solution is obtained.

$\Rightarrow$ all $\lambda_m{}^4$ must be positive.

$\int_0^l [(5) \times X_n - (6) \times X_m] dx \Rightarrow$

$\int_0^l (X_n X_m'''' - X_m X_n'''') dx = (\lambda_m{}^4 - \lambda_n{}^4) \int_0^l X_m X_n dx$

$= [X_n X_m''' - X_n' X_m'' - X_m X_n''' + X_m' X_n'']_0^l + \int_0^l (X_n'' X_m'' - X_m'' X_n'') dx = 0$

$\Rightarrow (\lambda_m{}^4 - \lambda_n{}^4) \int_0^l X_m X_n dx = 0$

$\Rightarrow$ eigenfucs corresponding to distinct eigenvalues are orthogonal.

Apply b.c$'s$ to (4):

$X_0(0) = 0 \Rightarrow c_1 = 0$

$X_0''(0) = 0 \Rightarrow c_3 = 0$

$$\left\{ \begin{array}{ll} X(l) = 0 & \Rightarrow \\ X'(l) = 0 & \Rightarrow \end{array} \right. = (\eta) \left\{ \begin{array}{l} \frac{c_2}{\lambda}(\sin \lambda l + \sinh \lambda l) + \frac{c_4}{\lambda^3}(\sin \lambda l - \sinh \lambda l) = 0 \\ \lambda c_2(-\sin \lambda l + \sinh \lambda l) + \frac{c_4}{\lambda}(-\sin \lambda l - \sinh \lambda l) = 0 \end{array} \right.$$

For nontrivical solution,

$$\begin{vmatrix} \frac{1}{\lambda}(\sin \lambda l + \sinh \lambda l) & \frac{1}{\lambda^3}(\sin \lambda l - \sinh \lambda l \\ \lambda(-\sin \lambda l + \sinh \lambda l & \frac{1}{\lambda}(-\sin \lambda l - \sinh \lambda l) \end{vmatrix} = 0$$

$\Rightarrow -\frac{4}{\lambda^2} \sin \lambda l \sinh \lambda l = 0$

since $\lambda^4 > 0 \Rightarrow \sinh \lambda l \neq 0$

$\Rightarrow \sin \lambda l = 0$

$\rightarrow \lambda_n^4 = (\frac{n\pi}{l})^4 \quad n = 1, 2....$

Thus $(\eta) \Rightarrow$

$$\frac{c_{2n}}{\lambda_n} \sinh \lambda_n l - \frac{c_{4n}}{\lambda_n^3} \sinh \lambda_n l = 0$$

$$\lambda_n c_{2n} \sinh \lambda_n l - \frac{c_{4n}}{\lambda_n} \sinh \lambda_n l = 0$$

from which $\Rightarrow c_{4n} = {\lambda_n}^2 c_{2n}$ then return to (4) $\Rightarrow$ the eigenfucs.

$$X_n(x) = \frac{2c_{2n}}{\lambda_n} \sin \lambda_n x$$

Sol. of $T_n(t)$:

$$T_n(t) = c_{1n} \cos c{\lambda_n}^2 t + c_{2n} \sin c{\lambda_n}^2 t$$

And then the general sol. for $v(x,t)$ is

$$v(x,t) = \sum_{n=1}^{\infty} (c_{1n} \cos c{\lambda_n}^2 t + c_{2n} \sin c{\lambda_n}^2 t) \sin \lambda_n x$$

For the I.C.'s:

$$v(x,0) = \sum_{n=1}^{\infty} c_{1n} \sin \lambda_n x = f(x)$$

$$\Rightarrow c_{1n} = \frac{2}{l} \int_0^l f(x) \sin \lambda_n x dx$$

$$v_t(x,0) = \sum_{n=1}^{\infty} c_{2n} c{\lambda_n}^2 \sin \lambda_n x = g(x)$$

$$\Rightarrow c_{2n} = \frac{2}{l} \cdot \frac{1}{c{\lambda_n}^2} \int_0^l g(x) \sin \lambda_n x dx$$