are normed linear spaces, and with the exception of Example 6 they are all Banach spaces.

Inner Product Spaces

A normed linear space provides us with a way of measuring the length of a vector. We want to refine the structure further, so that we shall also have available a notion of angle between vectors (in particular, we want to be able to tell whether or not two vectors are perpendicular). By analogy with ordinary three-dimensional vectors, we shall derive these notions from an inner product (also known as a scalar or dot product).

DEFINITION. An *inner product* $\langle x, y\rangle$ on a *real* linear space $\mathscr{A}^{(r)}$ is a *real*-valued function of ordered pairs of vectors x, y with the properties

$$\left.\begin{aligned}&\langle x, y\rangle = \langle y, x\rangle;\\ &\langle \alpha x, y\rangle = \alpha\langle x, y\rangle;\\ &\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle;\\ &\langle x, x\rangle \ge 0, \qquad \text{with } \langle x, x\rangle = 0 \text{ if and only if } x = 0.\end{aligned}\right\} \tag{2.6}$$

Theorem (*The Schwarz Inequality*). *For any two vectors x and y, we have*

$$\langle x, y\rangle^2 \le \langle x, x\rangle\langle y, y\rangle. \tag{2.7}$$

Proof. It follows from (2.6) that, for any real number α,

$$0 \le \langle x + \alpha y, x + \alpha y\rangle = \alpha^2\langle y, y\rangle + 2\alpha\langle x, y\rangle + \langle x, x\rangle.$$

The right side is a nonnegative quadratic in α, so that its discriminant is nonpositive; therefore,

$$\langle x, y\rangle^2 - \langle x, x\rangle\langle y, y\rangle \le 0,$$

which is the desired inequality.

Letting $\langle x, x\rangle^{1/2}$ and $\langle y, y\rangle^{1/2}$ stand for nonnegative square roots, we have

$$|\langle x, y\rangle| \le \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}. \tag{2.7a}$$

The reader should prove that the equality in (2.7) or in (2.7a) occurs if and only if x and y are dependent.

Using (2.7a),

$$\begin{aligned}\langle x + y, x + y\rangle &= \langle x, x\rangle + \langle y, y\rangle + 2\langle x, y\rangle\\ &\le \langle x, x\rangle + \langle y, y\rangle + 2\langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}\\ &= [\langle x, x\rangle^{1/2} + \langle y, y\rangle^{1/2}]^2.\end{aligned}$$

Hence

$$\langle x + y, x + y\rangle^{1/2} \le \langle x, x\rangle^{1/2} + \langle y, y\rangle^{1/2}. \tag{2.7b}$$

It is now clear that the quantity $\langle x, x\rangle^{1/2}$ is a norm on our linear space.

From (2.6) and (2.7b) one verifies that $\langle x, x\rangle^{1/2}$ satisfies the conditions (2.4) on $\|x\|$. Thus an inner product space has a *natural norm* defined by

$$\|x\| = \langle x, x\rangle^{1/2}. \tag{2.8}$$

Since a norm generates a natural metric, we have

$$d(x, y) = \|x - y\| = \langle x - y, x - y\rangle^{1/2}. \tag{2.9}$$

We now rewrite the Schwarz inequality (2.7a) in the more attractive form

$$|\langle x, y\rangle| \le \|x\|\,\|y\|. \tag{2.10}$$

DEFINITION. Two vectors x and y are said to be *orthogonal* (or perpendicular) if $\langle x, y\rangle = 0$. A set of vectors, each pair of which is orthogonal, is called an *orthogonal set*.

If two vectors x and y satisfy $\langle x, y\rangle = 0$, we have $\langle x + y, x + y\rangle = \langle x, x\rangle + \langle y, y\rangle$ or $\|x + y\|^2 = \|x\|^2 + \|y\|^2$, so that the Pythagorean theorem holds. This justifies the use of the word orthogonal for the property $\langle x, y\rangle = 0$.

One obtains without difficulty the proposition: *An orthogonal set of nonzero vectors is independent.*

DEFINITION. The set of vectors αy, where $y \ne 0$ and α runs through all real numbers, is called the *line* generated by y. The set of vectors αy, where $y \ne 0$ and $\alpha \ge 0$, is called the *positive half-line* generated by y.

There are two unit vectors (that is, vectors of unit length) lying on the line generated by y, $y/\|y\|$ and $-y/\|y\|$.

DEFINITION. The projection of x on the line generated by y is the vector $x_p = \langle x, e\rangle e$, where e is either unit vector lying on the line (either choice of e yields the same vector, x_p).

This projection x_p may be written in terms of y,

$$x_p = \frac{\langle x, y\rangle}{\|y\|^2}\, y. \tag{2.11}$$

Given a vector x and a vector $y \ne 0$, x can be decomposed in one and only one way as the sum of two vectors, the first lying on the line generated by y and the second perpendicular to this line (see Figure 2.3). We have specifically,

$$x = x_p + z, \qquad x_p = \frac{\langle x, y\rangle}{\|y\|^2}\, y, \qquad z = x - \frac{\langle x, y\rangle}{\|y\|^2}\, y, \qquad \langle z, y\rangle = 0. \tag{2.12}$$

The above decomposition is unique. If $x_p' + z'$ were another such decomposition, then we would have $0 = (x_p - x_p') + (z - z')$, where $x_p - x_p'$ is proportional to y and $z - z'$ is orthogonal to y; since the vectors $x_p - x_p'$ and $z - z'$ are orthogonal, they are either independent or else at least one of them must be zero. They cannot be independent, since their sum vanishes; therefore one of them is 0, hence the other must also be 0. This states that $x_p = x_p'$ and